Preface

This collection is intended for use with a Co-requisite course, pairing a concurrent MAT 1043 - Introduction to Math(Quantitative Reasoning) (TXCCN MAT 1332) with an NCBO (Non-Course Based Option) support course. UTSA's MAT 1043 courses utilize Dana Center math pathways materials. This book is designed to supplement the MAT materials and provide a free, opensource option for students to reference in the NCB support course.

The arrangement of materials within the modules is designed to support the Lessons covered in the MAT course. As a result the sequence may appear odd compared to a traditional developmental math textbook. It also follows our course sequence for NCB, which is a two day a week (50 minute) course that occurs on alternate days to the MAT course. (i.e. MAT 1043 may be MWF, and NCB TR).

This book utilizes topics from modules in OpenStax PreAlgebra, Intermediate Algebra, Introductory Statistics collections, and several other works found in the CNX repository. Each module cites the source where each section is derived from in footnotes, as well as when new content was created by one of the Authors.

This collection is intended for developmental students taking a Co-requisite course in math for a non-stem degree pathway.

Coverage and Scope

By design, the support course in our Co-requisite model emphasizes "Just-in-time" support of topics discussed in the related math course. It is not meant to be a complete review of all topics from earlier developmental math courses like Elementary or Intermediate Algebra. As a result, each chapter is arranged to review only those topics needed to understand concepts discussed in the specific math course chapter it supports. The material is presented as a sequence of modules tied to Lesson chapters of the MAT 1043 course. I.e. Module 2 supports Lessons 1A-D of MAT 1043. The order of topics was carefully planned to support progression throughout

the course (by providing foundational concepts related to the Lesson Topics, and/or previewing material for Lessons), and to facilitate a thorough understanding of each concept.

Chapter 1: Pre-knowledge Concepts
Chapter 2: Basics of Numbers, Decimals, Equations, Expressions and Election Math
These topics are aligned to Lessons 1A-D of MAT 1043.
Chapter 3: Data and Central tendency
These topics are aligned to Lessons 2A-2C of MAT 1043.
Chapter 4: Spread of Data and Percents
Chapter 4 covers Lessons 3A-3C of MAT 1043.
Chapter 5: Fractions, Ratios, and Probabilities
These topics provide review for Lessons 4A-4B of MAT 1043.
Chapter 6: Unit rates, Conversions, and Weighted Average
Chapter 6 reviews and supports Lessons 5A-6B
Chapter 7: Ratios
In Chapter 7, students can preview and find content supporting Lessons 7A-7F .
Chapter 8: Linear and Proportional Relationships
The topics in Chapter 8 support Lessons 8A-9A of MAT 1043.
Chapter 9: Linear Regression and Scatterplot Data
Students will find contnet supporting Lessons 9B-9D
Chapter 10: Exponential Growth
In Chapter 10, students find a brief review of Exponential growth for Lessons 10A-10B.
Chapter 11: Logistic Growth
Lessons 11A-11E are supported by Chaper 11.
Chapter 12: Sinusoidal Functions
In Chapter 12, students are introduced to content in Lessons 12A-12B of MAT 1043.

All chapters incorporate multiple topics, the titles of which can be viewed in the **Topics Covered** box at the start of each module.

About the Authors

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Errata

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Foundations

Topics to review prior to the first module. This is a blend of modules from Pre-algebra, Intermediate Algebra, and Elementary Algebra for use with the Corequisite QR math. By the end of this section, you will be able to:

- Use Variables and symbols
- Use Negatives and opposites
- Multiply and divide integers
- Use divisibility tests
- Find Factors
- Use the Rectangular Coordinate System

Note:

Topics Covered in this Module

In case you missed something in class, or just want to review a specific topic covered in this Module, here is a list of topics covered:

1. Use Variables and Algebraic Symbols [link]
2. Use Negatives and Opposites [link]
3. Multiply Integers [link]
4. Divide Integers [link]
5. Identify Multiples [link]
6. Use Common Divisibility Tests [link]
7. Find all the Factors of the Given Number [link]
8. Key Concepts [link]

Use Variables and Algebraic Symbols [footnote]

Section material derived from Openstax Prealgebra: The Language of Algebra-Use the Language of Algebra

Greg and Alex have the same birthday, but they were born in different years. This year Greg is 20 years old and Alex is 23, so Alex is 3 years older than Greg. When Greg was 12, Alex was 15. When Greg is 35, Alex will be 38. No matter what Greg's age is, Alex's age will always be 3 years more, right?

In the language of algebra, we say that Greg's age and Alex's age are variable and the three is a constant. The ages change, or vary, so age is a variable. The 3 years between them always stays the same, so the age difference is the constant.

In algebra, letters of the alphabet are used to represent variables. Suppose we call Greg's age g. Then we could use $g+3$ to represent Alex's age. See [link].

Greg's age	**Alex's age**
12	15
20	23
35	38

Greg's age	Alex's age
g	$g+3$

Letters are used to represent variables. Letters often used for variables are $x, y, a, b,$ and c.

Note:
A variable is a letter that represents a number or quantity whose value may change.
A constant is a number whose value always stays the same.

To write algebraically, we need some symbols as well as numbers and variables. There are several types of symbols we will be using. In Whole Numbers, we introduced the symbols for the four basic arithmetic operations: addition, subtraction, multiplication, and division. We will summarize them here, along with words we use for the operations and the result.

Operation	Notation	Say:	The result is…
Addition	$a+b$	a plus b	the sum of a and b
Subtraction	$a-b$	a minus b	the difference of a and b
Multiplication	$a\cdot b, (a)(b), (a)b, a(b)$	a times b	The product of a and b
Division	$a\div b, a/b, \frac{a}{b}, b\overline{)a}$	a divided by b	The quotient of a and b

In algebra, the cross symbol, $\times$, is not used to show multiplication because that symbol may cause confusion. Does $3xy$ mean $3 \times y$ (three times y) or $3\cdot x\cdot y$ (three times x times y)? To make it clear, use • or parentheses for multiplication.

We perform these operations on two numbers. When translating from symbolic form to words, or from words to symbolic form, pay attention to the words *of* or *and* to help you find the numbers.

- The *sum* ***of*** 5 ***and*** 3 means add 5 plus 3, which we write as $5+3$.
- The *difference* ***of*** 9 ***and*** 2 means subtract 9 minus 2, which we write as $9-2$.
- The *product* ***of*** 4 ***and*** 8 means multiply 4 times 8, which we can write as $4\cdot 8$.
- The *quotient* ***of*** 20 ***and*** 5 means divide 20 by 5, which we can write as $20\div 5$.

Example:
Translating Algebraic Expressions
Exercise:

Problem: Translate from algebra to words:

ⓐ $12+14$

ⓑ $(30)(5)$
ⓒ $64 \div 8$
ⓓ $x - y$

Solution:
Solution

ⓐ
$12 + 14$
12 plus 14
the sum of twelve and fourteen

ⓑ
$(30)(5)$
30 times 5
the product of thirty and five

ⓒ
$64 \div 8$
64 divided by 8
the quotient of sixty-four and eight

ⓓ
$x - y$

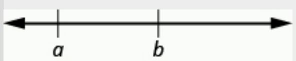
x minus y
the difference of x and y

When two quantities have the same value, we say they are equal and connect them with an *equal sign*.

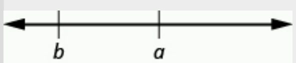
Note:
$a = b$ is read a is equal to b
The symbol $=$ is called the equal sign.

An inequality is used in algebra to compare two quantities that may have different values. The number line can help you understand inequalities. Remember that on the number line the numbers get larger as they go from left to right. So if we know that b is greater than a, it means that b is to the right of a on the number line. We use the symbols “$<$” and “$>$” for inequalities.

Note:
$a < b$ is read a is less than b
a is to the left of b on the number line



$a > b$ is read a is greater than b
a is to the right of b on the number line



The expressions $a < b$ and $a > b$ can be read from left-to-right or right-to-left, though in English we usually read from left-to-right. In general,
Equation:

$$\begin{array}{ll} a < b \text{ is equivalent to } b > a. & \text{For example, } 7 < 11 \text{ is equivalent to } 11 > 7. \\ a > b \text{ is equivalent to } b < a. & \text{For example, } 17 > 4 \text{ is equivalent to } 4 < 17. \end{array}$$

When we write an inequality symbol with a line under it, such as $a \leq b$, it means $a < b$ or $a = b$. We read this a is less than or equal to b. Also, if we put a slash through an equal sign, $\neq$, it means not equal.

We summarize the symbols of equality and inequality in [link].

Algebraic Notation	Say
$a = b$	a is equal to b
$a \neq b$	a is not equal to b
$a < b$	a is less than b
$a > b$	a is greater than b
$a \leq b$	a is less than or equal to b
$a \geq b$	a is greater than or equal to b

Note:
The symbols $<$ and $>$ each have a smaller side and a larger side.
smaller side $<$ larger side
larger side $>$ smaller side
The smaller side of the symbol faces the smaller number and the larger faces the larger number.

Example:
Translating Algebraic Ineqality Expressions
Exercise:

Problem: Translate from algebra to words:

ⓐ $20 \leq 35$
ⓑ $11 \neq 15 - 3$
ⓒ $9 > 10 \div 2$
ⓓ $x + 2 < 10$

Solution:
Solution

ⓐ
$20 \leq 35$
20 is less than or equal to 35

ⓑ

$11 \neq 15 - 3$

11 is not equal to 15 minus 3

ⓒ

$9 > 10 \div 2$

9 is greater than 10 divided by 2

ⓓ

$x + 2 < 10$

x plus 2 is less than 10

Example:
Using Inequality Symbols in Expressions and Equations
Exercise:

Problem:

The information in [link] compares the fuel economy in miles-per-gallon (mpg) of several cars. Write the appropriate symbol $=$, $<$, or $>$ in each expression to compare the fuel economy of the cars.

Car	Prius 	Mini Cooper 	Toyota Corolla 	Versa 	Honda Fit 
Fuel economy (mpg)	48	27	28	26	27

(credit: modification of work by Bernard Goldbach, Wikimedia Commons)

ⓐ MPG of Prius_____ MPG of Mini Cooper
ⓑ MPG of Versa_____ MPG of Fit
ⓒ MPG of Mini Cooper_____ MPG of Fit
ⓓ MPG of Corolla_____ MPG of Versa
ⓔ MPG of Corolla_____ MPG of Prius

Solution:
Solution

ⓐ	
	MPG of Prius____MPG of Mini Cooper
Find the values in the chart.	48____27
Compare.	48 > 27
	MPG of Prius > MPG of Mini Cooper

ⓑ	
	MPG of Versa____MPG of Fit
Find the values in the chart.	26____27
Compare.	26 < 27
	MPG of Versa < MPG of Fit

ⓒ	
	MPG of Mini Cooper____MPG of Fit
Find the values in the chart.	27____27
Compare.	27 = 27
	MPG of Mini Cooper = MPG of Fit

ⓓ	

	MPG of Corolla____MPG of Versa
Find the values in the chart.	28____26
Compare.	28 > 26
	MPG of Corolla > MPG of Versa

ⓔ	
	MPG of Corolla____MPG of Prius
Find the values in the chart.	28____48
Compare.	28 < 48
	MPG of Corolla < MPG of Prius

Grouping symbols in algebra are much like the commas, colons, and other punctuation marks in written language. They indicate which expressions are to be kept together and separate from other expressions. [link] lists three of the most commonly used grouping symbols in algebra.

Common Grouping Symbols	
parentheses	( )
brackets	[ ]
braces	{ }

Here are some examples of expressions that include grouping symbols. We will simplify expressions like these later in this section.

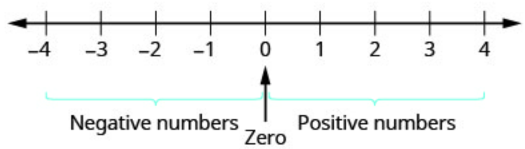
Equation:

$$8(14-8) \qquad 21-3[2+4(9-8)] \qquad 24\div\{13-2[1(6-5)+4]\}$$

Use Negatives and Opposites [footnote]

Section material derived from Openstax Prealgebra: Introduction to Integers-Introduction to Integers

Our work so far has only included the counting numbers and the whole numbers. But if you have ever experienced a temperature below zero or accidentally overdrawn your checking account, you are already familiar with negative numbers. **Negative numbers** are numbers less than 0. The negative numbers are to the left of zero on the number line. See [link].



The number line shows the location of positive and negative numbers.

The arrows on the ends of the number line indicate that the numbers keep going forever. There is no biggest positive number, and there is no smallest negative number.

Is zero a positive or a negative number? Numbers larger than zero are positive, and numbers smaller than zero are negative. Zero is neither positive nor negative.

Consider how numbers are ordered on the number line. Going from left to right, the numbers increase in value. Going from right to left, the numbers decrease in value. See [link].

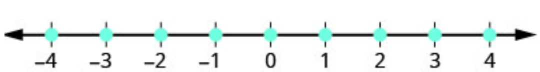
The numbers on a number line increase in value going from left to right and decrease in value going from right to left.

Remember that we use the notation:

$a < b$ (read "a is less than b") when a is to the left of b on the number line.

$a > b$ (read "*a* is greater than *b*") when *a* is to the right of *b* on the number line.

Now we need to extend the number line which showed the whole numbers to include negative numbers, too. The numbers marked by points in [link] are called the integers. The integers are the numbers $\ldots -3, -2, -1, 0, 1, 2, 3\ldots$



All the marked numbers are called *integers*.

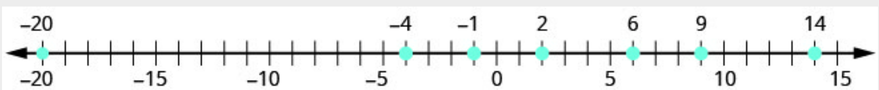
Example:
Numbers, Inequalities, and Number Lines
Exercise:

Problem:

Order each of the following pairs of numbers, using < or >: ⓐ $14\ \_\_\_\ 6$ ⓑ $-1\ \_\_\_\ 9$ ⓒ $-1\ \_\_\_\ -4$ ⓓ $2\ \_\_\_\ -20$.

Solution:
Solution

It may be helpful to refer to the number line shown.

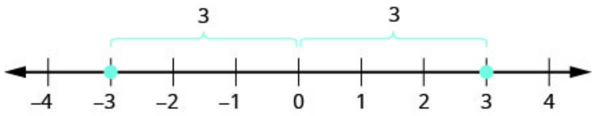


ⓐ 14 is to the right of 6 on the number line.	$14\ \_\_\_\ 6$ $14 > 6$
ⓑ −1 is to the left of 9 on the number line.	$-1\ \_\_\_\ 9$ $-1 < 9$
ⓒ −1 is to the right of −4 on the number line.	$-1\ \_\_\_\ -4$ $-1 > -4$
ⓓ 2 is to the right of −20 on the number line.	$2\ \_\_\_\ -20$ $2 > -20$

You may have noticed that, on the number line, the negative numbers are a mirror image of the positive numbers, with zero in the middle. Because the numbers 2 and -2 are the same distance from zero, they are called **opposites**. The opposite of 2 is -2, and the opposite of -2 is 2.

Note:
The **opposite** of a number is the number that is the same distance from zero on the number line but on the opposite side of zero.

[link] illustrates the definition.



The opposite of 3 is -3.

Sometimes in algebra the same symbol has different meanings. Just like some words in English, the specific meaning becomes clear by looking at how it is used. You have seen the symbol “−” used in three different ways.

Equation:

$10-4$ — Between two numbers, it indicates the operation of *subtraction*. We read $10-4$ as “10 minus 4.”

-8 — In front of a number, it indicates a *negative* number. We read -8 as “negative eight.”

$-x$ — In front of a variable, it indicates the *opposite*. We read $-x$ as “the opposite of x.”

$-(-2)$ — Here there are two “ − ” signs. The one in the parentheses tells us the number is negative 2. The one outside the parentheses tells us to take the *opposite* of -2. We read $-(-2)$ as “the opposite of negative two.”

$10-4$	Between two numbers, it indicates the operation of *subtraction*. We read $10-4$ as "10 minus 4."
-8	In front of a number, it indicates a *negative* number. We read −8 as "negative eight."
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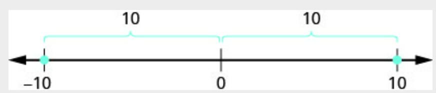
$-(-2)$	Here there are two "−" signs. The one in the parentheses tells us the number is negative 2. The one outside the parentheses tells us to take the *opposite* of −2. We read $-(-2)$ as "the opposite of negative two."

Note:
$-a$ means the opposite of the number a.
The notation $-a$ is read as “the opposite of a.”

Example:
Opposites on a Number Line
Exercise:

Problem: Find: ⓐ the opposite of 7 ⓑ the opposite of -10 ⓒ $-(-6)$.

Solution:
Solution

ⓐ -7 is the same distance from 0 as 7, but on the opposite side of 0.	 The opposite of 7 is -7.
ⓑ 10 is the same distance from 0 as -10, but on the opposite side of 0.	 The opposite of -10 is 10.
ⓒ $-(-6)$	 The opposite of $-(-6)$ is -6.

Our work with opposites gives us a way to define the integers.The whole numbers and their opposites are called the **integers**. The integers are the numbers $\ldots -3, -2, -1, 0, 1, 2, 3\ldots$

Note:
The whole numbers and their opposites are called the **integers**.
The integers are the numbers
Equation:

$$\ldots -3, -2, -1, 0, 1, 2, 3\ldots$$

When evaluating the opposite of a variable, we must be very careful. Without knowing whether the variable represents a positive or negative number, we don't know whether $-x$ is positive or negative. We can see this in [link].

Example:
Evaluating Algebraic Expressions

Exercise:

Problem: Evaluate ⓐ $-x$, when $x = 8$ ⓑ $-x$, when $x = -8$.

Solution:
Solution

ⓐ

To evaluate when $x = 8$ means to substitute 8 for x.	
	$-x$
Substitute 8 for x.	$-(8)$
Write the opposite of 8.	-8

ⓑ

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	$-x$
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Write the opposite of -8.	8

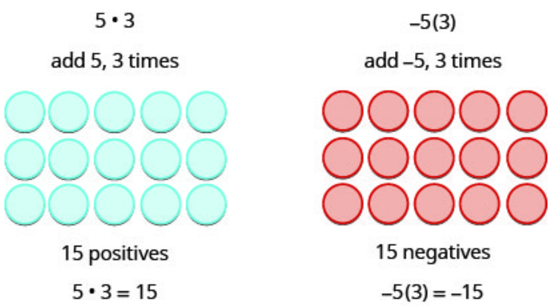
Multiply Integers [footnote]

Section material derived from Openstax Prealgebra: Introduction to Integers-Multiply and Divide Integers

Since multiplication is mathematical shorthand for repeated addition, our model can easily be applied to show multiplication of integers. Let's look at this concrete model to see what patterns we notice. We will use the same

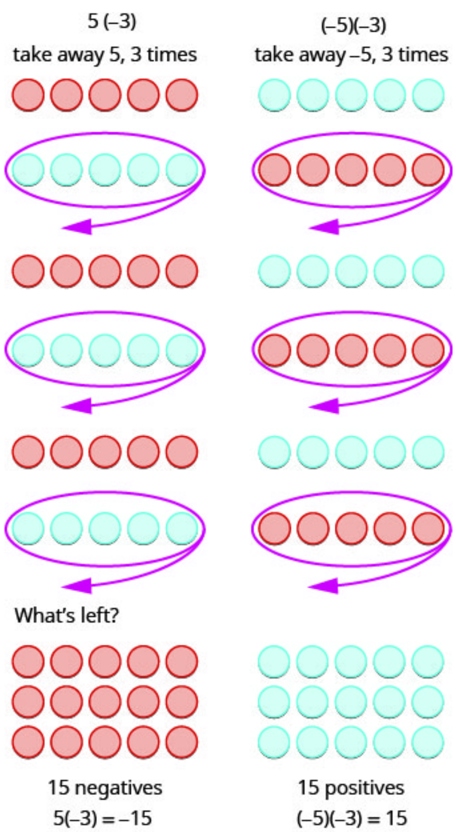
examples that we used for addition and subtraction. Here, we will use the model just to help us discover the pattern.

We remember that $a \cdot b$ means add a, b times. Here, we are using the model just to help us discover the pattern.



The next two examples are more interesting.

What does it mean to multiply 5 by -3? It means subtract 5, 3 times. Looking at subtraction as "taking away," it means to take away 5, 3 times. But there is nothing to take away, so we start by adding neutral pairs on the workspace. Then we take away 5 three times.



In summary:
Equation:

$$\begin{aligned} 5 \cdot 3 &= 15 & -5\,(3) &= -15 \\ 5\,(-3) &= -15 & (-5)\,(-3) &= 15 \end{aligned}$$

Notice that for multiplication of two signed numbers, when the:

- signs are the *same*, the product is *positive*.
- signs are *different*, the product is *negative*.

We'll put this all together in the chart below.

Note:
For multiplication of two signed numbers:

Same signs	**Product**	**Example**
Two positives Two negatives	Positive Positive	$7 \cdot 4 = 28$ $-8\left(-6\right) = 48$

Different signs	**Product**	**Example**
Positive · negative Negative · positive	Negative Negative	$7\left(-9\right) = -63$ $-5 \cdot 10 = -50$

Example:
Multiplying Numbers
Exercise:

Problem: Multiply: ⓐ $-9 \cdot 3$ ⓑ $-2\left(-5\right)$ ⓒ $4\left(-8\right)$ ⓓ $7 \cdot 6$.

Solution:
Solution

ⓐ Multiply, noting that the signs are different so the product is negative.	$-9 \cdot 3$ -27
ⓑ Multiply, noting that the signs are the same so the product is positive.	$-2\left(-5\right)$ 10

ⓒ Multiply, with different signs.	$4(-8)$ -32
ⓓ Multiply, with same signs.	$7 \cdot 6$ 42

When we multiply a number by 1, the result is the same number. What happens when we multiply a number by -1? Let's multiply a positive number and then a negative number by -1 to see what we get.

Equation:

	$-1 \cdot 4$	$-1(-3)$
Multiply.	-4	3
	-4 is the opposite of 4.	3 is the opposite of -3.

Each time we multiply a number by -1, we get its opposite!

Note:
Equation:

$$-1a = -a$$

Multiplying a number by -1 gives its opposite.

Example:
Multiplying Numbers
Exercise:

Problem: Multiply: ⓐ $-1 \cdot 7$ ⓑ $-1(-11)$.

Solution:
Solution

ⓐ Multiply, noting that the signs are different so the product is negative.	$-1 \cdot 7$ -7 -7 is the opposite of 7.
ⓑ Multiply, noting that the signs are the same so the product is positive.	$-1(-11)$ 11 11 is the opposite of -11.

Divide Integers [footnote]

Section material derived from Openstax Prealgebra: Introduction to Integers-Multiply and Divide Integers

What about division? Division is the inverse operation of multiplication. So, $15 \div 3 = 5$ because $15 \cdot 3 = 5$. In words, this expression says that 15 can be divided into three groups of five each because adding five three times gives 15. Look at some examples of multiplying integers, to figure out the rules for dividing integers.
Equation:

$$\begin{array}{rclcrcl} 5 \cdot 3 & = & 15 \text{ so } 15 \div 3 & = & 5 & \qquad -5\,(3) & = & -15 \text{ so } -15 \div 3 & = & -5 \\ (-5)\,(-3) & = & 15 \text{ so } 15 \div (-3) & = & -5 & \qquad 5\,(-3) & = & -15 \text{ so } -15 \div (-3) & = & 5 \end{array}$$

Division follows the same rules as multiplication!

For division of two signed numbers, when the:

- signs are the *same*, the quotient is *positive*.
- signs are *different*, the quotient is *negative*.

And remember that we can always check the answer of a division problem by multiplying.

Note:
For multiplication and division of two signed numbers:

- If the signs are the same, the result is positive.
- If the signs are different, the result is negative.

Same signs	Result
Two positives Two negatives	Positive Positive
If the signs are the same, the result is positive.	

Different signs	Result
Positive and negative Negative and positive	Negative Negative
If the signs are different, the result is negative.	

Example:
Dividing Numbers
Exercise:

Problem: Divide: ⓐ $-27 \div 3$ ⓑ $-100 \div (-4)$.

Solution:
Solution

ⓐ Divide. With different signs, the quotient is negative.	$-27 \div 3$ -9
ⓑ Divide. With signs that are the same, the quotient is positive.	$-100 \div (-4)$ 25

Identify Multiples of a Number [footnote]

Section material derived from Openstax Elementary Algebra: Introduction to the Language of Algebra-Find Multiples and Factors

The numbers 2, 4, 6, 8, 10, 12 are called multiples of 2. A **multiple** of 2 can be written as the product of a counting number and 2.

Multiples of 2:

2,	4,	6,	8,	10,	12, ...
$2 \cdot 1$	$2 \cdot 2$	$2 \cdot 3$	$2 \cdot 4$	$2 \cdot 5$	$2 \cdot 6$

Similarly, a multiple of 3 would be the product of a counting number and 3.

Multiples of 3:

3,	6,	9,	12,	15,	18, ...
$3 \cdot 1$	$3 \cdot 2$	$3 \cdot 3$	$3 \cdot 4$	$3 \cdot 5$	$3 \cdot 6$

We could find the multiples of any number by continuing this process.

Counting Number	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
Multiples of 2	2	4	6	8	10	12	14	16	18	20	22	24
Multiples of 3	3	6	9	12	15	18	21	24	27	30	33	36

Counting Number	1	2	3	4	5	6	7	8	9	10	11	12
Multiples of 4	4	8	12	16	20	24	28	32	36	40	44	48
Multiples of 5	5	10	15	20	25	30	35	40	45	50	55	60
Multiples of 6	6	12	18	24	30	36	42	48	54	60	66	72
Multiples of 7	7	14	21	28	35	42	49	56	63	70	77	84
Multiples of 8	8	16	24	32	40	48	56	64	72	80	88	96
Multiples of 9	9	18	27	36	45	54	63	72	81	90	99	108

Note:
A number is a **multiple** of n if it is the product of a counting number and n.

Another way to say that 15 is a multiple of 3 is to say that 15 is **divisible** by 3. That means that when we divide 3 into 15, we get a counting number. In fact, $15 \div 3$ is 5, so 15 is $5 \cdot 3$.

Use Common Divisibility Tests [footnote]

Section material derived from Openstax Prealgebra: Introduction to Whole Numbers-Introduction to Whole Numbers

Another way to say that 375 is a multiple of 5 is to say that 375 is divisible by 5. In fact, $375 \div 5$ is 75, so 375 is $5 \cdot 75$. Notice in [link] that $10{,}519$ is not a multiple 3. When we divided $10{,}519$ by 3 we did not get a counting number, so $10{,}519$ is not divisible by 3.

Note:
If a number m is a multiple of n, then we say that m is divisible by n.

Since multiplication and division are inverse operations, the patterns of multiples that we found can be used as divisibility tests. [link] summarizes divisibility tests for some of the counting numbers between one and ten.

Divisibility Tests	
A number is divisible by	
2	if the last digit is $0,\ 2,\ 4,\ 6,$ or 8

Divisibility Tests	
A number is divisible by	
3	if the sum of the digits is divisible by 3
5	if the last digit is 5 or 0
6	if divisible by both 2 and 3
10	if the last digit is 0

Example:
Divisibility Test
Exercise:

Problem: Determine whether 1,290 is divisible by $2, 3, 5,$ and 10.

Solution:
Solution

[link] applies the divisibility tests to 1,290. In the far right column, we check the results of the divisibility tests by seeing if the quotient is a whole number.

Divisible by…?	Test	Divisible?	Check
2	Is last digit $0, 2, 4, 6,$ or 8? *Yes.*	yes	$1290 \div 2 = 645$
3	Is sum of digits divisible by 3? $1 + 2 + 9 + 0 = 12$ *Yes.*	yes	$1290 \div 3 = 430$
5	Is last digit 5 or 0? *Yes.*	yes	$1290 \div 5 = 258$
10	Is last digit 0? *Yes.*	yes	$1290 \div 10 = 129$

Thus, 1,290 is divisible by $2, 3, 5,$ and 10.

Example:
Divisibility Test
Exercise:

Problem: Determine whether 5,625 is divisible by $2, 3, 5,$ and 10.

Solution:
Solution

[link] applies the divisibility tests to 5,625 and tests the results by finding the quotients.

Divisible by…?	Test	Divisible?	Check
2	Is last digit $0,\ 2,\ 4,\ 6,$ or 8? *No.*	no	$5625 \div 2 = 2812.5$
3	Is sum of digits divisible by 3? $5+6+2+5=18$ *Yes.*	yes	$5625 \div 3 = 1875$
5	Is last digit is 5 or 0? *Yes.*	yes	$5625 \div 5 = 1125$
10	Is last digit 0? *No.*	no	$5625 \div 10 = 562.5$

Thus, 5,625 is divisible by 3 and 5, but not 2, or 10.

Find all the Factors of the Given Number [footnote]

Section material derived from Openstax Prealgebra: Introduction to Whole Numbers-Find Multiples and Factors

There are often several ways to talk about the same idea. So far, we've seen that if m is a multiple of n, we can say that m is divisible by n. We know that 72 is the product of 8 and 9, so we can say 72 is a multiple of 8 and 72 is a multiple of 9. We can also say 72 is divisible by 8 and by 9. Another way to talk about this is to say that 8 and 9 are factors of 72. When we write $72 = 8 \cdot 9$ we can say that we have factored 72.

$$\underbrace{8 \cdot 9}_{\text{factors}} = \underbrace{72}_{\text{product}}$$

Note:
If $a \cdot b = m$, then a and b are factors of m, and m is the product of a and b.

In algebra, it can be useful to determine all of the factors of a number. This is called factoring a number, and it can help us solve many kinds of problems.

Note: Doing the Manipulative Mathematics activity "Model Multiplication and Factoring" will help you develop a better understanding of multiplication and factoring.

For example, suppose a choreographer is planning a dance for a ballet recital. There are 24 dancers, and for a certain scene, the choreographer wants to arrange the dancers in groups of equal sizes on stage.

In how many ways can the dancers be put into groups of equal size? Answering this question is the same as identifying the factors of 24. [link] summarizes the different ways that the choreographer can arrange the dancers.

Number of Groups	**Dancers per Group**	**Total Dancers**
1	24	$1 \cdot 24 = 24$
2	12	$2 \cdot 12 = 24$
3	8	$3 \cdot 8 = 24$
4	6	$4 \cdot 6 = 24$
6	4	$6 \cdot 4 = 24$
8	3	$8 \cdot 3 = 24$
12	2	$12 \cdot 2 = 24$
24	1	$24 \cdot 1 = 24$

What patterns do you see in [link]? Did you notice that the number of groups times the number of dancers per group is always 24? This makes sense, since there are always 24 dancers.

You may notice another pattern if you look carefully at the first two columns. These two columns contain the exact same set of numbers—but in reverse order. They are mirrors of one another, and in fact, both columns list all of the factors of 24, which are:
Equation:

$$1, 2, 3, 4, 6, 8, 12, 24$$

We can find all the factors of any counting number by systematically dividing the number by each counting number, starting with 1. If the quotient is also a counting number, then the divisor and the quotient are factors of the number. We can stop when the quotient becomes smaller than the divisor.

Note:

Divide the number by each of the counting numbers, in order, until the quotient is smaller than the divisor.

- If the quotient is a counting number, the divisor and quotient are a pair of factors.
- If the quotient is not a counting number, the divisor is not a factor.

List all the factor pairs.
Write all thefactorsin order from smallest to largest.

Example:
Finding Factors
Exercise:

Problem: Find all the factors of 72.

Solution:

Solution

Divide 72 by each of the counting numbers starting with 1. If the quotient is a whole number, the divisor and quotient are a pair of factors.

Dividend	Divisor	Quotient	Factors
72	1	72	1, 72
72	2	36	2, 36
72	3	24	3, 24
72	4	18	4, 18
72	5	14.4	–
72	6	12	6, 12
72	7	~10.29	–
72	8	9	8, 9

The next line would have a divisor of 9 and a quotient of 8. The quotient would be smaller than the divisor, so we stop. If we continued, we would end up only listing the same factors again in reverse order. Listing all the factors from smallest to greatest, we have

$1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, \text{and } 72$

Note:
Additional Online Resources

- Divisibility Rules
- Factors
- Ex 1: Determine Factors of a Number
- Ex 2: Determine Factors of a Number
- Ex 3: Determine Factors of a Number

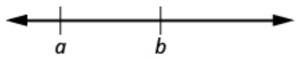
Key Concepts

- **Mathematical operations**

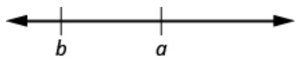
Operation	Notation	Say:	The result is…
Addition	$a+b$	a plus b	the sum of a and b
Multiplication	$a\cdot b, (a)(b), (a)b, a(b)$	a times b	The product of a and b
Subtraction	$a-b$	a minus b	the difference of a and b
Division	$a \div b, a/b, \frac{a}{b}, b\overline{)a}$	a divided by b	The quotient of a and b

- **Equality Symbol**
  - $a=b$ is read as a is equal to b
  - The symbol $=$ is called the equal sign.

- **Inequality**
  - $a < b$ is read a is less than b
  - a is to the left of b on the number line



  - $a > b$ is read a is greater than b
  - a is to the right of b on the number line



Algebraic Notation	Say
$a = b$	a is equal to b
$a \neq b$	a is not equal to b
$a < b$	a is less than b
$a > b$	a is greater than b
$a \leq b$	a is less than or equal to b
$a \geq b$	a is greater than or equal to b

- **Addition of Positive and Negative Integers**

$5 + 3$ 8 both positive, sum positive	$-5 + (-3)$ -8 both negative, sum negative
$-5 + 3$ -2 different signs, more negatives sum negative	$5 + (-3)$ 2 different signs, more positives sum positive

- **Property of Absolute Value**: $|n| \geq 0$ for all numbers. Absolute values are always greater than or equal to zero!

- **Subtraction of Integers**

$5-3$	$-5-(-3)$
2	-2
5 positives	5 negatives
take away 3 positives	take away 3 negatives
2 positives	2 negatives
$-5-3$	$5-(-3)$
-8	8
5 negatives, want to	5 positives, want to
subtract 3 positives	subtract 3 negatives
need neutral pairs	need neutral pairs

- **Subtraction Property:** Subtracting a number is the same as adding its opposite.
- **Factors** If $a \cdot b = m$, then a and b are factors of m, and m is the product of a and b.
- **Find all the factors of a counting number.**

  Divide the number by each of the counting numbers, in order, until the quotient is smaller than the divisor.

  a. If the quotient is a counting number, the divisor and quotient are a pair of factors.
  b. If the quotient is not a counting number, the divisor is not a factor.

  List all the factor pairs.
  Write all the factors in order from smallest to largest.

- **Determine if a number is prime.**

  Test each of the primes, in order, to see if it is a factor of the number.
  Start with 2 and stop when the quotient is smaller than the divisor or when a prime factor is found.
  If the number has a prime factor, then it is a composite number. If it has no prime factors, then the number is prime.

- **Divisibility tests**

Divisibility Tests	
A number is divisible by	
2	if the last digit is **0, 2, 4, 6,** or **8**
3	if the sum of the digits is divisible by **3**
5	if the last digit is **5** or **0**
6	if divisible by both **2** and **3**
10	if the last digit is **0**

- **Fraction Addition**

- If a, b, and c are numbers where $c \neq 0$, then $\frac{a}{c} + \frac{b}{c} = \frac{a+c}{c}$.
- To add fractions, add the numerators and place the sum over the common denominator.

- **Fraction Subtraction**
  - If a, b, and c are numbers where $c \neq 0$, then $\frac{a}{c} - \frac{b}{c} = \frac{a-c}{c}$.
  - To subtract fractions, subtract the numerators and place the difference over the common denominator.

- **Find the least common denominator (LCD) of two fractions.**

  Factor each denominator into its primes.
  List the primes, matching primes in columns when possible.
  Bring down the columns.
  Multiply the factors. The product is the LCM of the denominators.
  The LCM of the denominators is the LCD of the fractions.

- **Equivalent Fractions Property**
  - If a, b, and c are whole numbers where $b \neq 0$, $c \neq 0$ then $\frac{a}{b} = \frac{a \cdot c}{b \cdot c}$ and $\frac{a \cdot c}{b \cdot c} = \frac{a}{b}$

- **Convert two fractions to equivalent fractions with their LCD as the common denominator.**

  Find the LCD.
  For each fraction, determine the number needed to multiply the denominator to get the LCD.
  Use the Equivalent Fractions Property to multiply the numerator and denominator by the number from Step 2.
  Simplify the numerator and denominator.

- **Add or subtract fractions with different denominators.**

  Find the LCD.
  Convert each fraction to an equivalent form with the LCD as the denominator.
  Add or subtract the fractions.
  Write the result in simplified form.

- **Simplify complex fractions.**

  Simplify the numerator.
  Simplify the denominator.
  Divide the numerator by the denominator.
  Simplify if possible.

- **Sign Patterns of the Quadrants**

Quadrant I	**Quadrant II**	**Quadrant III**	**Quadrant IV**
(x,y)	(x,y)	(x,y)	(x,y)
$(+,+)$	$(-,+)$	$(-,-)$	$(+,-)$

- **Coordinates of Zero**

- Points with a y-coordinate equal to 0 are on the x-axis, and have coordinates (a, 0).
- Points with a x-coordinate equal to 0 are on the y-axis, and have coordinates (0, b).
- The point (0, 0) is called the origin. It is the point where the x-axis and y-axis intersect.

Glossary

Equation
: An equation is made up of two expressions connected by an equal sign.

Expression
: An expression is a number, a variable, or a combination of numbers and variables and operation symbols.

Ordered Pair
: combination of two numbers that describes the location of a point in the rectangular coordinate system in which the first number s the x-coordinate of the point, and the second number is the y-coordinate of the point.

Origin
: the point where the x-axis and y-axis intersect; coordinates (0,0).

Quadrant
: one of four regions into which the coordinate plane is divided by the axes.

Rectangular Coordinate System
: system of describing locations relative to a vertical and a horizontal axis.

x-axis
: the horizontal axis in a rectangular coordinate system.

y-axis
: the vertical axis on a rectangular coordinate system.

Numbers: Lessons 1.A-1.C
By the end of this section, you will be able to:

- Identify counting numbers and whole numbers
- Identify the place value of a digit
- Use Place Value to Name and Write Whole Numbers
- Round Whole Numbers

Note:
In case you missed something in class, or just want to review a specific topic covered in this Module, here is a list of topics covered:

1. Identify Counting Numbers and Whole Numbers [link]
2. Identify the Place Value of a Digit [link]
3. Use Place Value to Name Whole Numbers [link]
4. Use Place Value to Write Whole Numbers [link]
5. Round Whole Numbers [link]
6. Key Concepts [link]

Identify Counting Numbers and Whole Numbers [footnote]

Section material derived from Openstax Prealgebra: Introduction-Introduction to Whole Numbers

Learning algebra is similar to learning a language. You start with a basic vocabulary and then add to it as you go along. You need to practice often until the vocabulary becomes easy to you. The more you use the vocabulary, the more familiar it becomes.

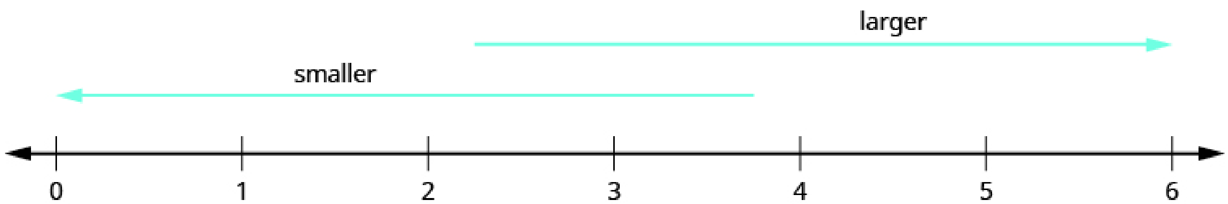
Algebra uses numbers and symbols to represent words and ideas. Let's look at the numbers first. The most basic numbers used in algebra are those we use to count objects: $1, 2, 3, 4, 5, \ldots$ and so on. These are called the **counting numbers**. The notation "…" is called an ellipsis, which is another

way to show “and so on”, or that the pattern continues endlessly. Counting numbers are also called natural numbers.

Note:
The counting numbers start with 1 and continue.
Equation:

$$1, 2, 3, 4, 5\ldots$$

Counting numbers and whole numbers can be visualized on a **number line** as shown in [link].



The point labeled 0 is called the **origin**. The points are equally spaced to the right of 0 and labeled with the counting numbers. When a number is paired with a point, it is called the **coordinate** of the point.

The discovery of the number zero was a big step in the history of mathematics. Including zero with the counting numbers gives a new set of numbers called the **whole numbers**.

Note:
The whole numbers are the counting numbers and zero.
Equation:

$$0, 1, 2, 3, 4, 5\ldots$$

We stopped at 5 when listing the first few counting numbers and whole numbers. We could have written more numbers if they were needed to make the patterns clear.

Example:
Identifying Counting and Whole Numbers
Exercise:

Problem:

Which of the following are ⓐ counting numbers? ⓑ whole numbers?

$0, \frac{1}{4}, 3, 5.2, 15, 105$

Solution:
Solution

ⓐ The counting numbers start at 1, so 0 is not a counting number. The numbers $3, 15,$ and 105 are all counting numbers.
ⓑ Whole numbers are counting numbers and 0. The numbers $0, 3, 15,$ and 105 are whole numbers.

The numbers $\frac{1}{4}$ and 5.2 are neither counting numbers nor whole numbers. We will discuss these numbers later.

Identify the Place Value of a Digit [footnote]

Section material derived from Openstax Prealgebra: Introduction-Introduction to Whole Numbers

By looking at money and base-10 blocks, we can see that each place in a number has a different value. A place value chart is a useful way to summarize this information. The place values are separated into groups of three, called periods. The periods are *ones, thousands, millions, billions, trillions*, and so on. In a written number, commas separate the periods.

Just as with the base-10 blocks, where the value of the tens rod is ten times the value of the ones block and the value of the hundreds square is ten times the tens rod, the value of each place in the place-value chart is ten times the value of the place to the right of it.

[link] shows how the number 5,278,194 is written in a place value chart.

Place Value														
Trillions			**Billions**			**Millions**			**Thousands**			**Ones**		
Hundred trillions	Ten trillions	Trillions	Hundred billions	Ten billions	Billions	Hundred millions	Ten millions	Millions	Hundred thousands	Ten thousands	Thousands	Hundreds	Tens	Ones
								5	2	7	8	1	9	4

- The digit 5 is in the millions place. Its value is 5,000,000.
- The digit 2 is in the hundred thousands place. Its value is 200,000.
- The digit 7 is in the ten thousands place. Its value is 70,000.
- The digit 8 is in the thousands place. Its value is 8,000.
- The digit 1 is in the hundreds place. Its value is 100.
- The digit 9 is in the tens place. Its value is 90.
- The digit 4 is in the ones place. Its value is 4.

Example:
Identifying Place Value
Exercise:

Problem:

In the number 63,407,218; find the place value of each of the following digits:

ⓐ 7
ⓑ 0
ⓒ 1
ⓓ 6
ⓔ 3

Solution:
Solution

Write the number in a place value chart, starting at the right.

Trillions			Billions			Millions			Thousands			Ones		
Hundred trillions	Ten trillions	Trillions	Hundred billions	Ten billions	Billions	Hundred millions	Ten millions	Millions	Hundred thousands	Ten thousands	Thousands	Hundreds	Tens	Ones
							6	3	4	0	7	2	1	8

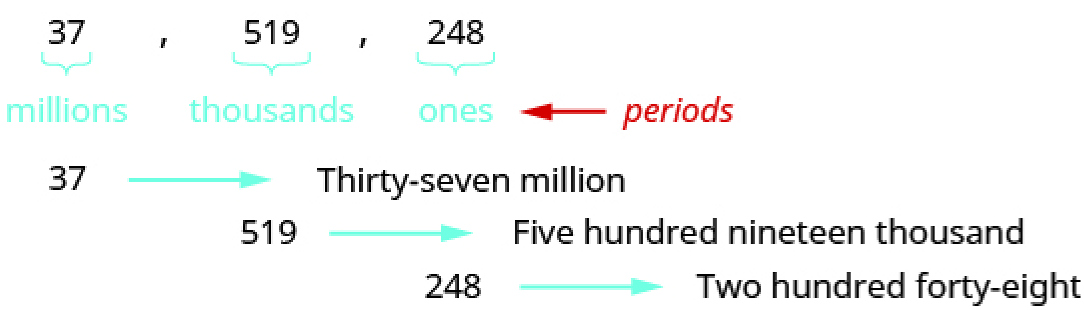
- ⓐ The 7 is in the thousands place.
- ⓑ The 0 is in the ten thousands place.
- ⓒ The 1 is in the tens place.

- ⓓ The 6 is in the ten millions place.
- ⓔ The 3 is in the millions place.

Use Place Value to Name Whole Numbers [footnote]

Section material derived from Openstax Prealgebra: Introduction-Introduction to Whole Numbers

When you write a check, you write out the number in words as well as in digits. To write a number in words, write the number in each period followed by the name of the period without the ‘s’ at the end. Start with the digit at the left, which has the largest place value. The commas separate the periods, so wherever there is a comma in the number, write a comma between the words. The ones period, which has the smallest place value, is not named.



So the number 37,519,248 is written thirty-seven million, five hundred nineteen thousand, two hundred forty-eight.

Notice that the word *and* is not used when naming a whole number.

Note:

Starting at the digit on the left, name the number in each period, followed

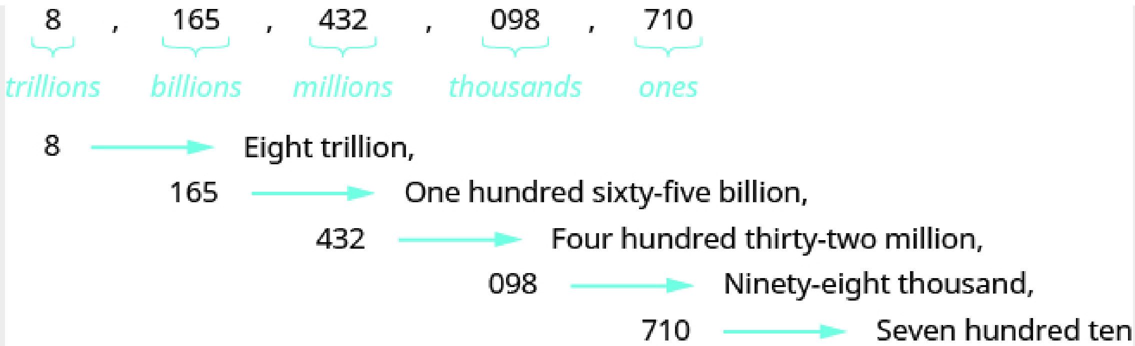
by the period name. Do not include the period name for the ones.
Use commas in the number to separate the periods.

Example:
Naming a Number
Exercise:

Problem: Name the number 8,165,432,098,710 in words.

Solution:
Solution

Begin with the leftmost digit, which is 8. It is in the trillions place.	eight trillion
The next period to the right is billions.	one hundred sixty-five billion
The next period to the right is millions.	four hundred thirty-two million
The next period to the right is thousands.	ninety-eight thousand
The rightmost period shows the ones.	seven hundred ten



Putting all of the words together, we write $8,165,432,098,710$ as eight trillion, one hundred sixty-five billion, four hundred thirty-two million, ninety-eight thousand, seven hundred ten.

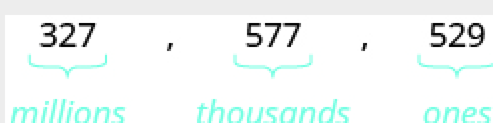
Example:
Naming a Number
Exercise:

Problem:

A student conducted research and found that the number of mobile phone users in the United States during one month in 2014 was $327,577,529$. Name that number in words.

Solution:
Solution

Identify the periods associated with the number.



Name the number in each period, followed by the period name. Put the commas in to separate the periods.

Millions period: three hundred twenty-seven million

Thousands period: five hundred seventy-seven thousand

Ones period: five hundred twenty-nine

So the number of mobile phone users in the Unites States during the month of April was three hundred twenty-seven million, five hundred seventy-seven thousand, five hundred twenty-nine.

Use Place Value to Write Whole Numbers [footnote]

Section material derived from Openstax Prealgebra: Introduction-Introduction to Whole Numbers

We will now reverse the process and write a number given in words as digits.

Note:

Identify the words that indicate periods. (Remember the ones period is never named.)
Draw three blanks to indicate the number of places needed in each period. Separate the periods by commas.
Name the number in each period and place the digits in the correct place value position.

Example:
Writing Numbers
Exercise:

Problem: Write the following numbers using digits.

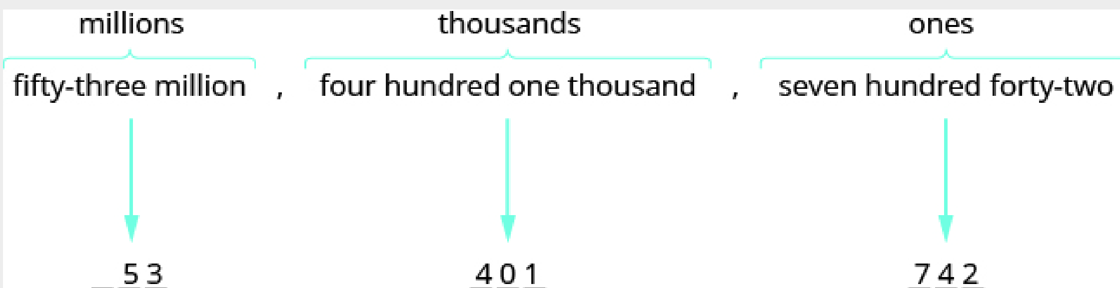
ⓐ fifty-three million, four hundred one thousand, seven hundred forty-two
ⓑ nine billion, two hundred forty-six million, seventy-three thousand, one hundred eighty-nine

Solution:
Solution

ⓐ Identify the words that indicate periods.

Except for the first period, all other periods must have three places. Draw three blanks to indicate the number of places needed in each period. Separate the periods by commas.

Then write the digits in each period.



Put the numbers together, including the commas. The number is $53{,}401{,}742$.

ⓑ Identify the words that indicate periods.

Except for the first period, all other periods must have three places. Draw three blanks to indicate the number of places needed in each period. Separate the periods by commas.

Then write the digits in each period.

The number is 9,246,073,189.

Notice that in part ⓑ, a zero was needed as a place-holder in the hundred thousands place. Be sure to write zeros as needed to make sure that each period, except possibly the first, has three places.

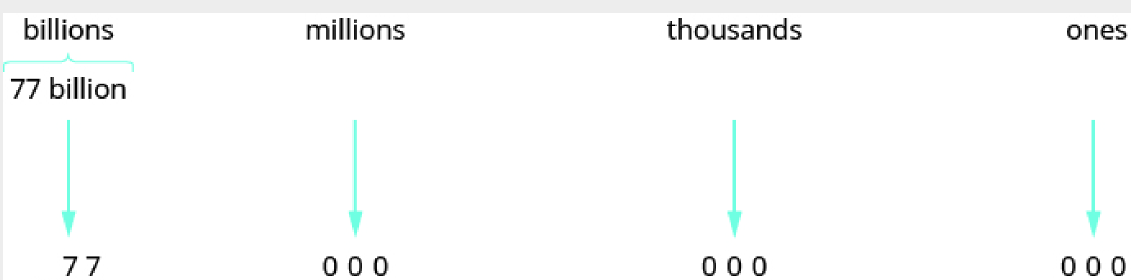
Example:
Writing Numbers
Exercise:

Problem:

A state budget was about $77 billion. Write the budget in standard form.

Solution:
Solution

Identify the periods. In this case, only two digits are given and they are in the billions period. To write the entire number, write zeros for all of the other periods.



So the budget was about $77,000,000,000.

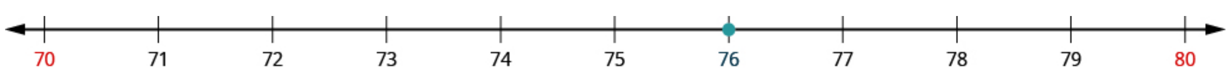
Round Whole Numbers [footnote]

Section material derived from Openstax Prealgebra: Introduction- Introduction to Whole Numbers

In 2013, the U.S. Census Bureau reported the population of the state of New York as 19,651,127 people. It might be enough to say that the population is approximately 20 million. The word *approximately* means that 20 million is not the exact population, but is close to the exact value.

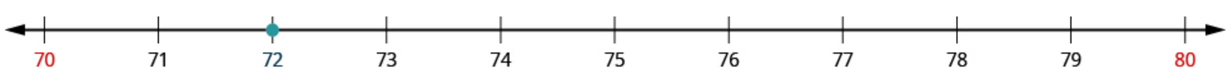
The process of approximating a number is called **rounding**. Numbers are rounded to a specific place value depending on how much accuracy is needed. 20 million was achieved by rounding to the millions place. Had we rounded to the one hundred thousands place, we would have 19,700,000 as a result. Had we rounded to the ten thousands place, we would have 19,650,000 as a result, and so on. The place value to which we round to depends on how we need to use the number.

Using the number line can help you visualize and understand the rounding process. Look at the number line in [link]. Suppose we want to round the number 76 to the nearest ten. Is 76 closer to 70 or 80 on the number line?



We can see that 76 is closer to 80 than to 70. So 76 rounded to the nearest ten is 80.

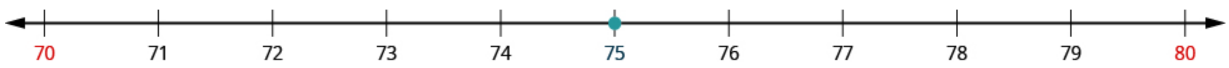
Now consider the number 72. Find 72 in [link].



We can see that 72 is closer to 70, so 72 rounded to the nearest ten is

70.

How do we round 75 to the nearest ten. Find 75 in [link].

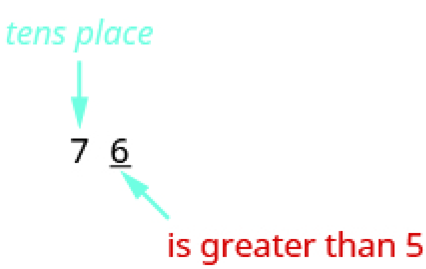


The number 75 is exactly midway between 70 and 80.

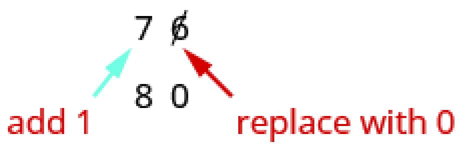
So that everyone rounds the same way in cases like this, mathematicians have agreed to round to the higher number, 80. So, 75 rounded to the nearest ten is 80.

Now that we have looked at this process on the number line, we can introduce a more general procedure. To round a number to a specific place, look at the number to the right of that place. If the number is less than 5, round down. If it is greater than or equal to 5, round up.

So, for example, to round 76 to the nearest ten, we look at the digit in the ones place.

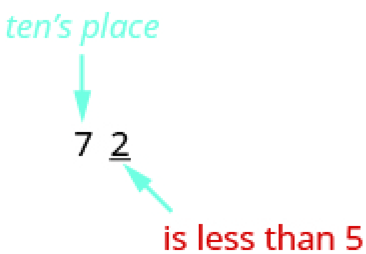


The digit in the ones place is a 6. Because 6 is greater than or equal to 5, we increase the digit in the tens place by one. So the 7 in the tens place becomes an 8. Now, replace any digits to the right of the 8 with zeros. So, 76 rounds to 80.

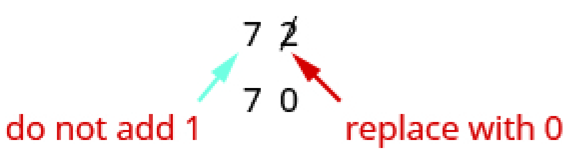


76 rounded to the nearest ten is 80.

Let’s look again at rounding 72 to the nearest 10. Again, we look to the ones place.



The digit in the ones place is 2. Because 2 is less than 5, we keep the digit in the tens place the same and replace the digits to the right of it with zero. So 72 rounded to the nearest ten is 70.



Note:

Locate the given place value. All digits to the left of that place value do not change.
Underline the digit to the right of the given place value.
Determine if this digit is greater than or equal to 5.

- Yes—add 1 to the digit in the given place value.
- No—do not change the digit in the given place value.

Replace all digits to the right of the given place value with zeros.

Example:
Rounding Numbers
Exercise:

Problem: Round 843 to the nearest ten.

Solution:
Solution

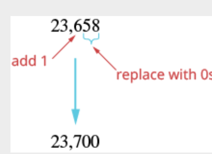
Locate the tens place.	
Underline the digit to the right of the tens place.	$84\underline{3}$
Since 3 is less than 5, do not change the digit in the tens place.	$84\underline{3}$
Replace all digits to the right of the tens place with zeros.	$84\underline{0}$
	Rounding 843 to the nearest ten gives 840.

Example:
Rounding Numbers
Exercise:

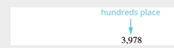
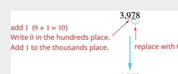
Problem: Round each number to the nearest hundred:

ⓐ 23,658
ⓑ 3,978

Solution:
Solution

ⓐ	
Locate the hundreds place.	
The digit of the right of the hundreds place is 5. Underline the digit to the right of the hundreds place.	23,6<u>5</u>8
Since 5 is greater than or equal to 5, round up by adding 1 to the digit in the hundreds place. Then replace all digits to the right of the hundreds place with zeros.	 So 23,658 rounded to the nearest

	hundred is 23,700.

ⓑ	
Locate the hundreds place.	
Underline the digit to the right of the hundreds place.	
The digit to the right of the hundreds place is 7. Since 7 is greater than or equal to 5, round up by added 1 to the 9. Then place all digits to the right of the hundreds place with zeros.	 So 3,978 rounded to the nearest hundred is 4,000.

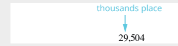
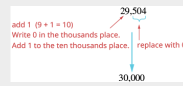
Example:
Rounding Numbers
Exercise:

Problem: Round each number to the nearest thousand:

ⓐ 147,032
ⓑ 29,504

Solution:
Solution

ⓐ	
Locate the thousands place. Underline the digit to the right of the thousands place.	
The digit to the right of the thousands place is 0. Since 0 is less than 5, we do not change the digit in the thousands place.	147,032
We then replace all digits to the right of the thousands pace with zeros.	147,032 So 147,032 rounded to the nearest thousand is 147,000.

ⓑ	
Locate the thousands place.	
Underline the digit to the right of the thousands place.	29,504
The digit to the right of the thousands place is 5. Since 5 is greater than or equal to 5, round up by adding 1 to the 9. Then replace all digits to the right of the thousands place with zeros.	 So 29,504 rounded to the nearest thousand is 30,000.

Notice that in part ⓑ, when we add 1 thousand to the 9 thousands, the total is 10 thousands. We regroup this as 1 ten thousand and 0 thousands. We add the 1 ten thousand to the 3 ten thousands and put a 0 in the thousands place.

Note:
ACCESS ADDITIONAL ONLINE RESOURCES

- Determine Place Value
- Write a Whole Number in Digits from Words

Key Concepts

Naming Whole Numbers

Place Value														
Trillions			Billions			Millions			Thousands			Ones		
Hundred trillions	Ten trillions	Trillions	Hundred billions	Ten billions	Billions	Hundred millions	Ten millions	Millions	Hundred thousands	Ten thousands	Thousands	Hundreds	Tens	Ones
								5	2	7	8	1	9	4

- **Name a whole number in words.**

  Starting at the digit on the left, name the number in each period, followed by the period name. Do not include the period name for the ones.
  Use commas in the number to separate the periods.

- **Use place value to write a whole number.**

  Identify the words that indicate periods. (Remember the ones period is never named.)
  Draw three blanks to indicate the number of places needed in each period.
  Name the number in each period and place the digits in the correct place value position.

- **Round a whole number to a specific place value.**

  Locate the given place value. All digits to the left of that place value do not change.
  Underline the digit to the right of the given place value.

Determine if this digit is greater than or equal to 5. If yes—add 1 to the digit in the given place value. If no—do not change the digit in the given place value.
Replace all digits to the right of the given place value with zeros.

Glossary

Counting Numbers
: The counting numbers are the numbers 1, 2, 3, ….

Number Line
: A number line is used to visualize numbers. The numbers on the number line get larger as they go from left to right, and smaller as they go from right to left.

Place Value system
: Our number system is called a place value system because the value of a digit depends on its position, or place, in a number.

Rounding
: The process of approximating a number is called rounding.

Whole Numbers
: The whole numbers are the numbers 0, 1, 2, 3, ….

Decimals: Lessons 1.A - 1.C
This module works with Lessons 1A-1D of the corequisite course for MAT 1043 (Quantitative Reasoning).

Note:
In case you missed something in class, or just want to review a specific topic covered in this Module, here is a list of topics covered:

1. Name Decimals [link]
2. Write Decimals [link]
3. Round Decimals [link]
4. Repeating Decimals [link]
5. Decimal Operations [link]
6. Convert Decimals to Fraction or Mixed Numbers [link]
7. Key Concepts [link]

Name Decimals [footnote]

Section material derived from Openstax Prealgebra: Introduction to Decimals-Decimals

You probably already know quite a bit about decimals based on your experience with money. Suppose you buy a sandwich and a bottle of water for lunch. If the sandwich costs $\$3.45$, the bottle of water costs $\$1.25$, and the total sales tax is $\$0.33$, what is the total cost of your lunch?

$3.45	Sandwich
$1.25	Water
+ $0.33	Tax
$5.03	Total

The total is $\$5.03$. Suppose you pay with a $\$5$ bill and 3 pennies. Should you wait for change? No, $\$5$ and 3 pennies is the same as $\$5.03$.

Because 100 pennies $= \$1$, each penny is worth $\frac{1}{100}$ of a dollar. We write the value of one penny as $\$0.01$, since $0.01 = \frac{1}{100}$.

Writing a number with a decimal is known as decimal notation. It is a way of showing parts of a whole when the whole is a power of ten. In other words, decimals are another way of writing fractions whose denominators are powers of ten. Just as the counting numbers are based on powers of ten, decimals are based on powers of ten. [link] shows the counting numbers.

Counting number	Name
1	One
$10 = 10$	Ten
$10 \cdot 10 = 100$	One hundred
$10 \cdot 10 \cdot 10 = 1000$	One thousand
$10 \cdot 10 \cdot 10 \cdot 10 = 10{,}000$	Ten thousand

How are decimals related to fractions? [link] shows the relation.

Decimal	Fraction	Name
0.1	$\frac{1}{10}$	One tenth

Decimal	Fraction	Name
0.01	$\frac{1}{100}$	One hundredth
0.001	$\frac{1}{1,000}$	One thousandth
0.0001	$\frac{1}{10,000}$	One ten-thousandth

When we name a whole number, the name corresponds to the place value based on the powers of ten. In Whole Numbers, we learned to read 10,000 as *ten thousand*. Likewise, the names of the decimal places correspond to their fraction values. Notice how the place value names in [link] relate to the names of the fractions from [link].

Place Value											
Hundred thousands	Ten thousands	Thousands	Hundreds	Tens	Ones		Tenths	Hundredths	Thousandths	Ten-thousandths	Hundred-thousandths
						.					

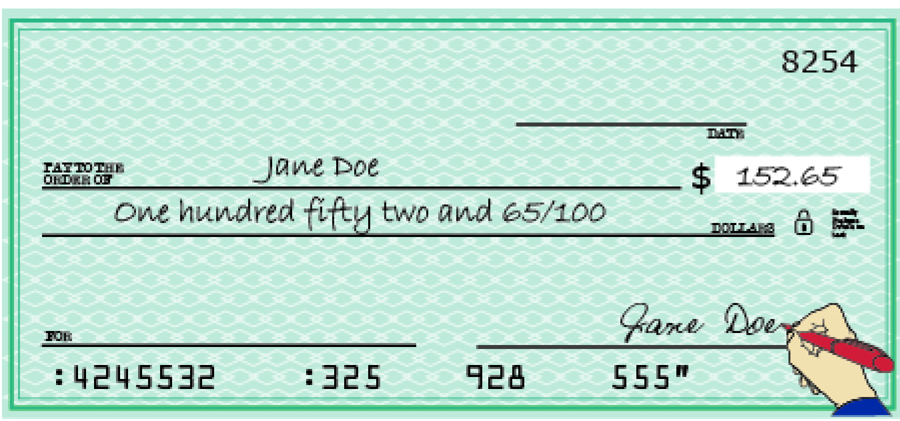
This chart illustrates place values to the left and right of the decimal point.

Notice two important facts shown in [link].

- The “th” at the end of the name means the number is a fraction. “One thousand” is a number larger than one, but “one thousandth” is a number smaller than one.
- The tenths place is the first place to the right of the decimal, but the tens place is two places to the left of the decimal.

Remember that $\$5.03$ lunch? We read $\$5.03$ as *five dollars and three cents*. Naming decimals (those that don’t represent money) is done in a similar way. We read the number 5.03 as *five and three hundredths*.

We sometimes need to translate a number written in decimal notation into words. As shown in [link], we write the amount on a check in both words and numbers.



When we write a check, we write the amount as a decimal number as well as in words. The bank looks at the check to make sure both numbers match. This helps prevent errors.

Let's try naming a decimal, such as 15.68.	
We start by naming the number to the left of the decimal.	fifteen______
We use the word "and" to indicate the decimal point.	fifteen and_____
Then we name the number to the right of the decimal point as if it were a whole number.	fifteen and sixty-eight_____
Last, name the decimal place of the last digit.	fifteen and sixty-eight hundredths

The number 15.68 is read *fifteen and sixty-eight hundredths*.

Note:

- Name the number to the left of the decimal point.
- Write "and" for the decimal point.
- Name the "number" part to the right of the decimal point as if it were a whole number.
- Name the decimal place of the last digit.

Example:
Naming Decimal Numbers
Exercise:

Problem: Name each decimal: ⓐ 4.3 ⓑ 2.45 ⓒ 0.009 ⓓ -15.571.

Solution:
Solution

ⓐ	
	4.3
Name the number to the left of the decimal point.	four_____
Write "and" for the decimal point.	four and_____
Name the number to the right of the decimal point as if it were a whole number.	four and three_____
Name the decimal place of the last digit.	four and three tenths

ⓑ	
	2.45
Name the number to the left of the decimal point.	two_____

Write "and" for the decimal point.	two and_____
Name the number to the right of the decimal point as if it were a whole number.	two and forty-five_____
Name the decimal place of the last digit.	two and forty-five hundredths

ⓒ	
	0.009
Name the number to the left of the decimal point.	Zero is the number to the left of the decimal; it is not included in the name.
Name the number to the right of the decimal point as if it were a whole number.	nine_____
Name the decimal place of the last digit.	nine thousandths

ⓓ	

	-15.571
Name the number to the left of the decimal point.	negative fifteen
Write "and" for the decimal point.	negative fifteen and_____
Name the number to the right of the decimal point as if it were a whole number.	negative fifteen and five hundred seventy-one_____
Name the decimal place of the last digit.	negative fifteen and five hundred seventy-one thousandths

Write Decimals [footnote]

Section material derived from Openstax Prealgebra: Introduction to Decimals-Decimals

Now we will translate the name of a decimal number into decimal notation. We will reverse the procedure we just used.

Let’s start by writing the number six and seventeen hundredths:

	six and seventeen hundredths

The word *and* tells us to place a decimal point.	___.___
The word before *and* is the whole number; write it to the left of the decimal point.	6._____
The decimal part is seventeen hundredths. Mark two places to the right of the decimal point for hundredths.	6._ _
Write the numerals for seventeen in the places marked.	6.17

Example:
Writing Decimal Numbers
Exercise:

Problem: Write fourteen and thirty-seven hundredths as a decimal.

Solution:
Solution

	fourteen and thirty-seven hundredths
Place a decimal point under the word 'and'.	______. _________
Translate the words before 'and' into the whole number and place it to the	14. _________

left of the decimal point.	
Mark two places to the right of the decimal point for “hundredths”.	14.__ __
Translate the words after “and” and write the number to the right of the decimal point.	14.37
	Fourteen and thirty-seven hundredths is written 14.37.

Note:

Look for the word “and”—it locates the decimal point.

Mark the number of decimal places needed to the right of the decimal point by noting the place value indicated by the last word.

- Place a decimal point under the word “and.” Translate the words before “and” into the whole number and place it to the left of the decimal point.
- If there is no “and,” write a “0” with a decimal point to its right.

Translate the words after “and” into the number to the right of the decimal point. Write the number in the spaces—putting the final digit in the last place.

Fill in zeros for place holders as needed.

The second bullet in Step 2 is needed for decimals that have no whole number part, like ‘nine thousandths’. We recognize them by the words that indicate the place value after the decimal – such as ‘tenths’ or ‘hundredths.’ Since there is no whole number, there is no ‘and.’ We start by placing a zero to the left of the decimal and continue by filling in the numbers to the right, as we did above.

Example:
Writing Decimal Numbers
Exercise:

Problem: Write twenty-four thousandths as a decimal.

Solution:
Solution

	twenty-four thousandths
Look for the word "and".	There is no "and" so start with 0 0.
To the right of the decimal point, put three decimal places for thousandths.	0. _ _ _ tenths hundredths thousandths
Write the number 24 with the 4 in the thousandths place.	0. _ 2 4 tenths hundredths thousandths

Put zeros as placeholders in the remaining decimal places.	0.024
	So, twenty-four thousandths is written 0.024

Before we move on to our next objective, think about money again. We know that $\$1$ is the same as $\$1.00$. The way we write $\$1$ (or $\$1.00$) depends on the context. In the same way, integers can be written as decimals with as many zeros as needed to the right of the decimal.

Equation:

$$\begin{array}{ll} 5 = 5.0 & -2 = -2.0 \\ 5 = 5.00 & -2 = -2.00 \\ 5 = 5.000 & -2 = -2.000 \end{array}$$

Equation:

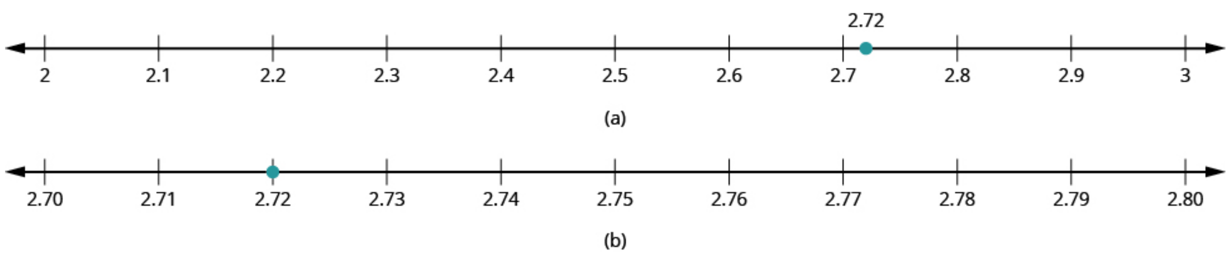
$$\text{and so on}\ldots$$

Round Decimals [footnote]

Section material derived from Openstax Prealgebra: Introduction to Decimals-Decimals

In the United States, gasoline prices are usually written with the decimal part as thousandths of a dollar. For example, a gas station might post the price of unleaded gas at $\$3.279$ per gallon. But if you were to buy exactly one gallon of gas at this price, you would pay $\$3.28$, because the final price would be rounded to the nearest cent. In Whole Numbers, we saw that we round numbers to get an approximate value when the exact value is not

needed. Suppose we wanted to round $\$2.72$ to the nearest dollar. Is it closer to $\$2$ or to $\$3$? What if we wanted to round $\$2.72$ to the nearest ten cents; is it closer to $\$2.70$ or to $\$2.80$? The number lines in [link] can help us answer those questions.



ⓐ We see that 2.72 is closer to 3 than to 2. So, 2.72 rounded to the nearest whole number is 3.
ⓑ We see that 2.72 is closer to 2.70 than 2.80. So we say that 2.72 rounded to the nearest tenth is 2.7.

Can we round decimals without number lines? Yes! We use a method based on the one we used to round whole numbers.

Note:

Locate the given place value and mark it with an arrow.
Underline the digit to the right of the given place value.
Is this digit greater than or equal to 5?

- Yes - add 1 to the digit in the given place value.
- No - do not change the digit in the given place value

Rewrite the number, removing all digits to the right of the given place value.

Example:
Rounding Decimals
Exercise:

Problem: Round 18.379 to the nearest hundredth.

Solution:
Solution

	18.379
Locate the hundredths place and mark it with an arrow.	hundredths place ↓ 18.379
Underline the digit to the right of the 7.	hundredths place ↓ $18.37\underline{9}$
Because 9 is greater than or equal to 5, add 1 to the 7.	$18.37\underbrace{9}_{\text{delete}}$ add 1
Rewrite the number, deleting all	18.38

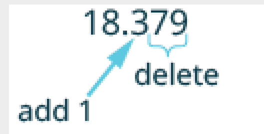
digits to the right of the hundredths place.	
	18.38 is 18.379 rounded to the nearest hundredth.

Example:
Rounding Decimals
Exercise:

Problem: Round 18.379 to the nearest ⓐ tenth ⓑ whole number.

Solution:
Solution

ⓐ Round 18.379 to the nearest tenth.	
	18.379
Locate the tenths place and mark it with an arrow.	tenths place 18.379

Underline the digit to the right of the tenths digit.	
Because 7 is greater than or equal to 5, add 1 to the 3.	
Rewrite the number, deleting all digits to the right of the tenths place.	18.4
	So, 18.379 rounded to the nearest tenth is 18.4.

ⓑ Round 18.379 to the nearest whole number.	
	18.379
Locate the ones place and mark it with an arrow.	

Underline the digit to the right of the ones place.	
Since 3 is not greater than or equal to 5, do not add 1 to the 8.	
Rewrite the number, deleting all digits to the right of the ones place.	18
	So 18.379 rounded to the nearest whole number is 18.

Note:
ACCESS ADDITIONAL ONLINE RESOURCES

- Introduction to Decimal Notation
- Write a Number in Decimal Notation from Words
- Identify Decimals on the Number Line
- Rounding Decimals
- Writing a Decimal as a Simplified Fraction

Repeating Decimals [footnote]

Section material derived from Openstax Prealgebra: Introduction to Decimals-Decimal Operations

So far, in all the examples converting fractions to decimals the division resulted in a remainder of zero. This is not always the case. Let's see what happens when we convert the fraction $\frac{4}{3}$ to a decimal. First, notice that $\frac{4}{3}$ is an improper fraction. Its value is greater than 1. The equivalent decimal will also be greater than 1.

We divide 4 by 3.

```
  1.333...
3)4.000
  3
  10
   9
   10
    9
    10
     9
     1
```

No matter how many more zeros we write, there will always be a remainder of 1, and the threes in the quotient will go on forever. The number $1.333...$ is called a repeating decimal. Remember that the "…" means that the pattern repeats.

> **Note:**
> A **repeating decimal** is a decimal in which the last digit or group of digits repeats endlessly.

How do you know how many 'repeats' to write? Instead of writing $1.333\ldots$ we use a shorthand notation by placing a line over the digits that repeat. The repeating decimal $1.333\ldots$ is written $1.\overline{3}$. The line above the 3 tells you that the 3 repeats endlessly. So $1.333... = 1.\overline{3}$

For other decimals, two or more digits might repeat. Table 12 shows some more examples of repeating decimals.

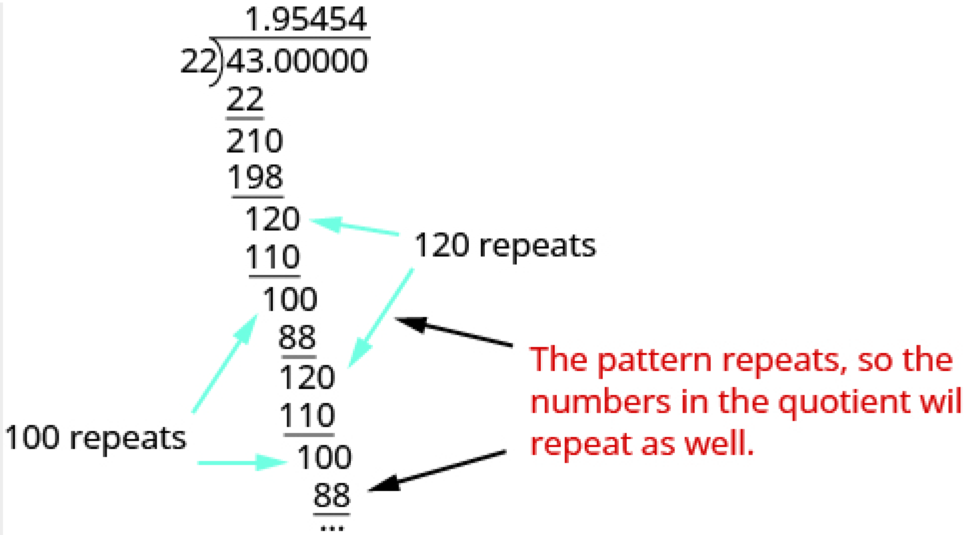
$1.333\ldots = 1.\overline{3}$	3 is the repeating digit
$4.1666... = 4.1\overline{6}$	6 is the repeating digit
$4.161616\ldots = 4.\overline{16}$	16 is the repeating digit
$0.271271271\ldots = 0.\overline{271}$	271 is the repeating digit

Example:
Writing a Repeating Decimal
Exercise:

Problem: Write $\frac{43}{22}$ as a decimal.

Solution:
Solution

Divide 43 by 22.



Notice that the differences of 120 and 100 repeat, so there is a repeat in the digits of the quotient; 54 will repeat endlessly. The first decimal place in the quotient, 9, is not part of the pattern. So,

Equation:

$$\frac{43}{22} = 1.9\overline{54}$$

It is useful to convert between fractions and decimals when we need to add or subtract numbers in different forms. To add a fraction and a decimal, for example, we would need to either convert the fraction to a decimal or the decimal to a fraction.

Example:
Adding a Fraction and a Decimal
Exercise:

Problem: Simplify: $\frac{7}{8} + 6.4$.

Solution:
Solution

		$\frac{7}{8}+6.4$
Change $\frac{7}{8}$ to a decimal.	$\begin{array}{r} 0.875 \\ 8\overline{)7.000} \\ \underline{64} \\ 60 \\ \underline{56} \\ 40 \\ \underline{40} \\ 0 \end{array}$	$0.875+6.4$
Add.		7.275

Decimal Operations [footnote]

Section material derived from Openstax Prealgebra: Introduction to Decimas-Decimal Operations

Note:

Write the numbers vertically so the decimal points line up.
Use zeros as place holders, as needed.
Add or subtract the numbers as if they were whole numbers. Then place

the decimal in the answer under the decimal points in the given numbers.

Example:
Adding Decimals
Exercise:

Problem: Add: $23.5 + 41.38$.

Solution:
Solution

	$23.5 + 41.38$
Write the numbers vertically so the decimal points line up.	$\begin{array}{r} 23.5 \\ +\ 41.38 \\ \hline \end{array}$
Place 0 as a place holder after the 5 in 23.5, so that both numbers have two decimal places.	$\begin{array}{r} 23.5\textcolor{red}{0} \\ +\ 41.38 \\ \hline \end{array}$
Add the numbers as if they were whole numbers. Then place the decimal in the answer under the decimal points in the given numbers.	$\begin{array}{r} 23.50 \\ +\ 41.38 \\ \hline 64.88 \end{array}$

Note:

Determine the sign of the product.
Write the numbers in vertical format, lining up the numbers on the right.
Multiply the numbers as if they were whole numbers, temporarily ignoring the decimal points.
Place the decimal point. The number of decimal places in the product is the sum of the number of decimal places in the factors. If needed, use zeros as placeholders.
Write the product with the appropriate sign.

Example:
Multiplying Decimals
Exercise:

Problem: Multiply: $(3.9)(4.075)$.

Solution:
Solution

	$(3.9)(4.075)$
Determine the sign of the product. The signs are the same.	The product will be positive.
Write the numbers in vertical format, lining up the numbers on the right.	$\begin{array}{r} 4.075 \\ \underline{\times\ 3.9} \end{array}$

Multiply the numbers as if they were whole numbers, temporarily ignoring the decimal points.	$\begin{array}{r} 4.075 \\ \times\, 3.9 \\ \hline 36675 \\ 12225 \\ \hline 158925 \end{array}$
Place the decimal point. Add the number of decimal places in the factors $(1 + 3\ )$. Place the decimal point 4 places from the right.	$\begin{array}{rl} 4.075 & \text{3 places} \\ \times\, 3.9 & \text{1 place} \\ \hline 36675 & \\ 12225 & \\ \hline 15\underbrace{8925} & \text{4 places} \end{array}$
The product is positive.	$(3.9)(4.075) = 15.8925$

Convert Decimals to Fractions or Mixed Numbers [footnote]

Section material derived from Openstax Prealgebra Introduction to Decimals-Decimals

We often need to rewrite decimals as fractions or mixed numbers. Let's go back to our lunch order to see how we can convert decimal numbers to fractions. We know that $\$5.03$ means 5 dollars and 3 cents. Since there are 100 cents in one dollar, 3 cents means $\frac{3}{100}$ of a dollar, so $0.03 = \frac{3}{100}$.

We convert decimals to fractions by identifying the place value of the farthest right digit. In the decimal 0.03, the 3 is in the hundredths place, so 100 is the denominator of the fraction equivalent to 0.03.
Equation:

$$0.03 = \frac{3}{100}$$

For our $\$5.03$ lunch, we can write the decimal 5.03 as a mixed number.
Equation:

$$5.03 = 5\frac{3}{100}$$

Notice that when the number to the left of the decimal is zero, we get a proper fraction. When the number to the left of the decimal is not zero, we get a mixed number.

Note:

Look at the number to the left of the decimal.

- If it is zero, the decimal converts to a proper fraction.
- If it is not zero, the decimal converts to a mixed number.
  - Write the whole number.

Determine the place value of the final digit.

Write the fraction.

- numerator—the ‘numbers’ to the right of the decimal point
- denominator—the place value corresponding to the final digit

Simplify the fraction, if possible.

Example:
Convert a decimal number to a fraction or mixed number.
Exercise:

Problem:

Write each of the following decimal numbers as a fraction or a mixed number:

ⓐ 4.09 ⓑ 3.7 ⓒ −0.286

Solution:
Solution

ⓐ	
	4.09
There is a 4 to the left of the decimal point. Write "4" as the whole number part of the mixed number.	$4\frac{\square}{\square}$
Determine the place value of the final digit.	4. 0 (tenths) 9 (hundredths)
Write the fraction. Write 9 in the numerator as it is the number to the right of the decimal point.	$4\frac{9}{\square}$
Write 100 in the denominator as the place value of the final digit, 9, is hundredth.	$4\frac{9}{100}$

The fraction is in simplest form.	So, 4.09 = $4\frac{9}{100}$

Did you notice that the number of zeros in the denominator is the same as the number of decimal places?

ⓑ	
	3.7
There is a 3 to the left of the decimal point. Write "3" as the whole number part of the mixed number.	$3\frac{\square}{\square}$
Determine the place value of the final digit.	3. 7 tenths
Write the fraction. Write 7 in the numerator as it is the number to the right of the decimal point.	$3\frac{7}{\square}$
Write 10 in the denominator as the place value of the final digit, 7, is tenths.	$3\frac{7}{10}$
The fraction is in simplest form.	So, 3.7 = $3\frac{7}{10}$

C	
	-0.286
There is a 0 to the left of the decimal point. Write a negative sign before the fraction.	$-\frac{\square}{\square}$
Determine the place value of the final digit and write it in the denominator.	$-0.\underset{\text{tenths}}{2}\ \underset{\text{hundredths}}{8}\ \underset{\text{thousandths}}{6}$
Write the fraction. Write 286 in the numerator as it is the number to the right of the decimal point. Write 1,000 in the denominator as the place value of the final digit, 6, is thousandths.	$-\frac{286}{1000}$
We remove a common factor of 2 to simplify the fraction.	$-\frac{143}{500}$

Key Concepts

Decimal Place values

Place Value											
Hundred thousands	Ten thousands	Thousands	Hundreds	Tens	Ones		Tenths	Hundredths	Thousandths	Ten-thousandths	Hundred-thousandths
						•					

- **Name a decimal number.**

  Name the number to the left of the decimal point.
  Write “and” for the decimal point.
  Name the “number” part to the right of the decimal point as if it were a whole number.
  Name the decimal place of the last digit.

- **Write a decimal number from its name.**

  Look for the word “and”—it locates the decimal point.
  Place a decimal point under the word “and.” Translate the words before “and” into the whole number and place it to the left of the decimal point.
  If there is no “and,” write a “0” with a decimal point to its right.
  Mark the number of decimal places needed to the right of the decimal point by noting the place value indicated by the last word.
  Translate the words after “and” into the number to the right of the decimal point. Write the number in the spaces—putting the final digit in the last place.
  Fill in zeros for place holders as needed.

- **Round a decimal.**

Locate the given place value and mark it with an arrow.
Underline the digit to the right of the given place value.

Is this digit greater than or equal to 5?	Yes - add 1 to the digit in the given place value.	No - do not change the digit in the given place value

Rewrite the number, removing all digits to the right of the given place value.

- **Convert a decimal number to a fraction or mixed number.**

Look at the number to the left of the decimal.	If it is zero, the decimal converts to a proper fraction.	If it is not zero, the decimal converts to a mixed number.	Write the whole number.

Determine the place value of the final digit.
Write the fraction. numerator—the ‘numbers’ to the right of the decimal point denominator—the place value corresponding to the final digit
Simplify the fraction, if possible.

- **Add or Subtract Decimals**

Write the numbers vertically so the decimal points line up.
Use zeros as place holders, as needed.
Add or subtract the numbers as if they were whole numbers. Then place the decimal in the answer under the decimal points in the given numbers.

- **Multiplying Decimal Numbers**
  - Determine the sign of the product.
  - Write the numbers in vertical format, lining up the numbers on the right.
  - Multiply the numbers as if they were whole numbers, temporarily ignoring the decimal points.
  - Place the decimal point. The number of decimal places in the product is the sum of the number of decimal places in the factors. If needed, use zeros as placeholders.
  - Write the product with the appropriate sign.

Glossary

Number Line
: A number line is used to visualize numbers. The numbers on the number line get larger as they go from left to right, and smaller as they go from right to left.

Place Value system
: Our number system is called a place value system because the value of a digit depends on its position, or place, in a number.

Rounding
: The process of approximating a number is called rounding.

Equivalent Decimals
: Two decimals are equivalent decimals if they convert to equivalent fractions.

Expressions: Lessons 1.A - 1.C

Note:
In case you missed something in class, or just want to review a specific topic covered in this Module, here is a list of topics covered:

1. Simplify Expressions using Order of Operations [link]
2. Identify terms, Coefficients, Like Terms [link]
3. Evaluate Algebraic Expressions [link]
4. Simplify Expressions by Combining Like Terms [link]
5. Identify Expressions and Equations [link]
6. Key Concepts [link]

Simplify Expressions Using the Order of Operations [footnote]

Section material derived from Openstax Prealgebra: The Language of Algebra-Use the Language of Algebra

We've introduced most of the symbols and notation used in algebra, but now we need to clarify the order of operations. Otherwise, expressions may have different meanings, and they may result in different values.

For example, consider the expression:
Equation:

$$4+3\cdot 7$$

Equation:

Some students say it simplifies to 49.		Some students say it simplifies to 25.	
	$4+3\cdot 7$		$4+3\cdot 7$
Since $4+3$ gives 7.	$7\cdot 7$	Since $3\cdot 7$ is 21.	$4+21$
And $7\cdot 7$ is 49.	49	And $21+4$ makes 25.	25

Imagine the confusion that could result if every problem had several different correct answers. The same expression should give the same result. So mathematicians established some guidelines called the order of operations, which outlines the order in which parts of an expression must be simplified.

Note:
When simplifying mathematical expressions perform the operations in the following order:
1. **P**arentheses and other Grouping Symbols

- Simplify all expressions inside the parentheses or other grouping symbols, working on the innermost parentheses first.

2. **E**xponents

- Simplify all expressions with exponents.

3. **M**ultiplication and **D**ivision

- Perform all multiplication and division in order from left to right. These operations have equal priority.

4. **A**ddition and **S**ubtraction

- Perform all addition and subtraction in order from left to right. These operations have equal priority.

This process is also known as **P.E.M.D.A.S.**.

Students often ask, “How will I remember the order?” Here is a way to help you remember: Take the first letter of each key word and substitute the silly phrase. **P**lease **E**xcuse **M**y **D**ear **A**unt **S**ally.

Order of Operations	
Please	**P**arentheses
Excuse	**E**xponents
My **D**ear	**M**ultiplication and **D**ivision
Aunt **S**ally	**A**ddition and **S**ubtraction

It’s good that ‘**M**y **D**ear’ goes together, as this reminds us that **m**ultiplication and **d**ivision have equal priority. We do not always do multiplication before division or always do

division before multiplication. We do them in order from left to right.

Similarly, '**A**unt **S**ally' goes together and so reminds us that **a**ddition and **s**ubtraction also have equal priority and we do them in order from left to right.

Note: Doing the Manipulative Mathematics activity Game of 24 will give you practice using the order of operations.

Example:
Order of Operations
Exercise:

Problem: Simplify:

ⓐ $18 \div 9 \cdot 2$
ⓑ $18 \cdot 9 \div 2$

Solution:
Solution

ⓐ	
	$18 \div 9 \cdot 2$
Are there any **p**arentheses? No.	
Are there any **e**xponents? No.	
Is there any **m**ultiplication or **d**ivision? Yes.	
Multiply and divide from left to right. Divide.	$2 \cdot 2$
Multiply.	

	4

ⓑ	
	$18 \cdot 9 \div 2$
Are there any **p**arentheses? No.	
Are there any **e**xponents? No.	
Is there any **m**ultiplication or **d**ivision? Yes.	
Multiply and divide from left to right.	
Multiply.	$162 \div 2$
Divide.	81

Example:
Order of Operations
Exercise:

Problem: Simplify: $18 \div 6 + 4(5 - 2)$.

Solution:
Solution

	$18 \div 6 + 4(5 - 2)$
Parentheses? Yes, subtract first.	$18 \div 6 + 4(3)$
Exponents? No.	
Multiplication or division? Yes.	
Divide first because we multiply and divide left to right.	$3 + 4(3)$
Any other multiplication or division? Yes.	
Multiply.	$3 + 12$
Any other multiplication or division? No.	
Any addition or subtraction? Yes.	15

When there are multiple grouping symbols, we simplify the innermost parentheses first and work outward.

Example:
Order of Operations
Exercise:

Problem: Simplify: $5 + 2^3 + 3\left[6 - 3(4 - 2)\right]$.

Solution:
Solution

	$5+2^3+3\,[6-3(4-2)].$
Are there any parentheses (or other grouping symbol)? Yes.	
Focus on the parentheses that are inside the brackets.	$5+2^3+3\,[6-3(4-2)].$
Subtract.	$5+2^3+3\,[6-3(2)].$
Continue inside the brackets and multiply.	$5+2^3+3\,[6-6].$
Continue inside the brackets and subtract.	$5+2^3+3\,[0].$
The expression inside the brackets requires no further simplification.	
Are there any exponents? Yes.	
Simplify exponents.	$5+2^3+3\,[0].$
Is there any multiplication or division? Yes.	
Multiply.	$5+8+3\,[0].$
Is there any addition or subtraction? Yes.	
Add.	$5+8+0.$
Add.	$13+0.$
	$13.$

Example:
Order of Operations
Exercise:

Problem: Simplify: $2^3+3^4\div 3-5^2.$

Solution:

Solution

	$2^3 + 3^4 \div 3 - 5^2.$
If an expression has several exponents, they may be simplified in the same step.	
Simplify exponents.	$2^3 + 3^4 \div 3 - 5^2.$
Divide.	$8 + 81 \div 3 - 25.$
Add.	$8 + 27 - 25.$
Subtract.	$35 - 25.$
	$10.$

Note:
ACCESS ADDITIONAL ONLINE RESOURCES

- Order of Operations
- Order of Operations – The Basics
- Ex: Evaluate an Expression Using the Order of Operations
- Example 3: Evaluate an Expression Using The Order of Operations

Identify Terms, Coefficients, and Like Terms [footnote]

Section material derived from Openstax Prealgebra: The Language of Algebra-Evaluate Simplify and Translate Expressions

Algebraic expressions are made up of *terms*. A **term** is a constant or the product of a constant and one or more variables. Some examples of terms are 7, y, $5x^2$, $9a$, and $13xy$.

The constant that multiplies the variable(s) in a term is called the **coefficient**. We can think of the coefficient as the number *in front of* the variable. The coefficient of the term $3x$ is 3. When we write x, the coefficient is 1, since $x = 1 \cdot x$. [link] gives the coefficients for each of the terms in the left column.

Term	Coefficient
7	7
$9a$	9
y	1
$5x^2$	5

Example:
Identifying Terms, Coefficients, and Like Terms
Exercise:

Problem:

Identify each term in the expression $9b + 15x^2 + a + 6$. Then identify the coefficient of each term.

Solution:
Solution

The expression has four terms. They are $9b$, $15x^2$, a, and 6.

The coefficient of $9b$ is 9.

The coefficient of $15x^2$ is 15.

Remember that if no number is written before a variable, the coefficient is 1. So the coefficient of a is 1.

The coefficient of a constant is the constant, so the coefficient of 6 is 6.

Some terms share common traits. Look at the following terms. Which ones seem to have traits in common?
Equation:

$$5x, 7, n^2, 4, 3x, 9n^2$$

Which of these terms are like terms?

- The terms 7 and 4 are both constant terms.
- The terms $5x$ and $3x$ are both terms with x.
- The terms n^2 and $9n^2$ both have n^2.

Terms are called **like terms** if they have the same variables and exponents. All constant terms are also like terms. So among the terms $5x, 7, n^2, 4, 3x, 9n^2$,
Equation:

$$7 \text{ and } 4 \text{ are like terms.}$$

Equation:

$$5x \text{ and } 3x \text{ are like terms.}$$

Equation:

$$n^2 \text{ and } 9n^2 \text{ are like terms.}$$

Note:
Terms that are either constants or have the same variables with the same exponents are like **terms**.

Example:
Identifying Like terms
Exercise:

Problem: Identify the like terms:

ⓐ $y^3, 7x^2, 14, 23, 4y^3, 9x, 5x^2$
ⓑ $4x^2 + 2x + 5x^2 + 6x + 40x + 8xy$

Solution: Solution

ⓐ $y^3, 7x^2, 14, 23, 4y^3, 9x, 5x^2$

Look at the variables and exponents. The expression contains y^3, x^2, x, and constants.

The terms y^3 and $4y^3$ are like terms because they both have y^3.

The terms $7x^2$ and $5x^2$ are like terms because they both have x^2.

The terms 14 and 23 are like terms because they are both constants.

The term $9x$ does not have any like terms in this list since no other terms have the variable x raised to the power of 1.

ⓑ $4x^2 + 2x + 5x^2 + 6x + 40x + 8xy$

Look at the variables and exponents. The expression contains the terms $4x^2, 2x, 5x^2, 6x, 40x, \text{and } 8xy$

The terms $4x^2$ and $5x^2$ are like terms because they both have x^2.

The terms $2x, 6x, \text{and } 40x$ are like terms because they all have x.

The term $8xy$ has no like terms in the given expression because no other terms contain the two variables xy.

Evaluate Algebraic Expressions [footnote]

Section material derived from Openstax Prealgebra: The Language of Algebra-Evaluate Simplify and Translate Expressions

In the last section, we simplified expressions using the order of operations. In this section, we'll evaluate expressions—again following the order of operations.

To **evaluate** an algebraic expression means to find the value of the expression when the variable is replaced by a given number. To evaluate an expression, we substitute the given number for the variable in the expression and then simplify the expression using the order of operations.

Example:
Evaluating Algebraic Expressions
Exercise:

Problem: Evaluate $x + 7$ when

ⓐ $x = 3$
ⓑ $x = 12$

Solution:
Solution

ⓐ To evaluate, substitute 3 for x in the expression, and then simplify.

	$x + 7$
Substitute.	$3 + 7$
Add.	10

When $x = 3$, the expression $x + 7$ has a value of 10.

ⓑ To evaluate, substitute 12 for x in the expression, and then simplify.

	$x + 7$
Substitute.	$12 + 7$
Add.	19

When $x = 12$, the expression $x + 7$ has a value of 19.

Notice that we got different results for parts ⓐ and ⓑ even though we started with the same expression. This is because the values used for x were different. When we evaluate an expression, the value varies depending on the value used for the variable.

Example:
Evaluating Algebraic Expressions
Exercise:

Problem: Evaluate $9x - 2$, when

ⓐ $x = 5$
ⓑ $x = 1$

Solution:
Solution

Remember ab means a times b, so $9x$ means 9 times x.

ⓐ To evaluate the expression when $x = 5$, we substitute 5 for x, and then simplify.

	$9x - 2$
Substitute 5 **for** x	$9*5 - 2$
Multiply.	$45 - 2$
Subtract.	43

ⓑ To evaluate the expression when $x = 1$, we substitute 1 for x, and then simplify.

	$9x - 2$
Substitute 1 **for** x	$9(1) - 2$
Multiply.	$9 - 2$
Subtract.	7

Notice that in part ⓐ that we wrote $9 \cdot 5$ and in part ⓑ we wrote $9(1)$. Both the dot and the parentheses tell us to multiply.

Example:
Evaluating Algebraic Expressions
Exercise:

Problem: Evaluate x^2 when $x = 10$.

Solution:
Solution

We substitute 10 for x, and then simplify the expression.

	x^2
Substitute 10 **for x**	10^2
Use the definition of exponent.	10*10
Multiply.	100

When $x = 10$, the expression x^2 has a value of 100.

Example:
Evaluating Algebraic Expressions
Exercise:

Problem: Evaluate 2^x when $x = 5$.

Solution:
Solution

In this expression, the variable is an exponent.

	2^x
Substitute 5 for x.	2^5
Use the definition of exponent.	$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$
Multiply.	32

When $x = 5$, the expression 2^x has a value of 32.

Example:
Evaluating Algebraic Expressions
Exercise:

Problem: Evaluate $3x + 4y - 6$ when $x = 10$ and $y = 2$.

Solution:
Solution

This expression contains two variables, so we must make two substitutions.

	$3x + 4y - 6$
Substitute 10 for *x* and 2 for *y*.	$3(10) + 4(2) - 6$
Multiply.	$30 + 8 - 6$

Add and subtract left to right.	32

When $x=10$ and $y=2$, the expression $3x+4y-6$ has a value of 32.

Example:
Evaluating Algebraic Expressions
Exercise:

Problem: Evaluate $2x^2+3x+8$ when $x=4$.

Solution:
Solution

We need to be careful when an expression has a variable with an exponent. In this expression, $2x^2$ means $2\cdot x\cdot x$ and is different from the expression $(2x)^2$, which means $2x\cdot 2x$.

	$2x^2+3x+8$
Substitute 4 **for each x**	$2(4)^2+3(4)+8$
Simplify 4^2.	$2(16)+3(4)+8$
Multiply.	$32+12+8$
Add.	52

Simplify Expressions by Combining Like Terms [footnote]

Section material derived from Openstax Prealgebra: The Language of Algebra-Evaluate Simplify and Translate Expressions

We can simplify an expression by combining the like terms. What do you think $3x + 6x$ would simplify to? If you thought $9x$, you would be right!.

We can see why this works by writing both terms as addition problems.

Insert image here

Add the coefficients and keep the same variable. It doesn't matter what x is. If you have 3 of something and add 6 more of the same thing, the result is 9 of them. For example, 3 oranges plus 6 oranges is 9 oranges. We will discuss the mathematical properties behind this later.

The expression $3x + 6x$ has only two terms. When an expression contains more terms, it may be helpful to rearrange the terms so that like terms are together. The Commutative Property of Addition says that we can change the order of addends without changing the sum. So we could rearrange the following expression before combining like terms.

Insert image here

Now it is easier to see the like terms to be combined.

Note:

Identify like terms.
Rearrange the expression so like terms are together.
Add the coefficients of the like terms.

Example:
Combining Like Terms
Exercise:

Problem: Simplify the expression: $3x + 7 + 4x + 5$.

Solution: Solution

	$3x + 7 + 4x + 5$

Identify the like terms.	$3x + 7 + 4x + 5$
Rearrange the expression, so the like terms are together.	$3x + 4x + 7 + 5$
Add the coefficients of the like terms.	$\underbrace{3x + 4x}_{7x} + \underbrace{7 + 5}_{12}$
The original expression is simplified to...	$7\text{x} + 12$

Identify Expressions and Equations [footnote]

Section material derived from Openstax Prealgebra: The Language of Algebra-Use the Language of Algebra

What is the difference in English between a phrase and a sentence? A phrase expresses a single thought that is incomplete by itself, but a sentence makes a complete statement. "Running very fast" is a phrase, but "The football player was running very fast" is a sentence. A sentence has a subject and a verb.

In algebra, we have *expressions* and *equations*. An expression is like a phrase. Here are some examples of expressions and how they relate to word phrases:

Expression	Words	Phrase
$3 + 5$	3 plus 5	the sum of three and five
$n - 1$	n minus one	the difference of n and one
$6 \cdot 7$	6 times 7	the product of six and seven
$\frac{x}{y}$	x divided by y	the quotient of x and y

Notice that the phrases do not form a complete sentence because the phrase does not have a verb. An equation is two expressions linked with an equal sign. When you read the words the symbols represent in an equation, you have a complete sentence in English. The equal sign gives the verb. Here are some examples of equations:

Equation	**Sentence**
$3+5=8$	The sum of three and five is equal to eight.
$n-1=14$	n minus one equals fourteen.
$6\cdot 7=42$	The product of six and seven is equal to forty-two.
$x=53$	x is equal to fifty-three.
$y+9=2y-3$	y plus nine is equal to two y minus three.

Note:
An **expression** is a number, a variable, or a combination of numbers and variables and operation symbols.
An **equation** is made up of two expressions connected by an equal sign.

Example:
Identifying Expressions and Equations
Exercise:

Problem: Determine if each is an expression or an equation:

ⓐ $16-6=10$
ⓑ $4\cdot 2+1$
ⓒ $x\div 25$
ⓓ $y+8=40$

Solution:
Solution

ⓐ $16 - 6 = 10$	This is an equation—two expressions are connected with an equal sign.
ⓑ $4 \cdot 2 + 1$	This is an expression—no equal sign.
ⓒ $x \div 25$	This is an expression—no equal sign.
ⓓ $y + 8 = 40$	This is an equation—two expressions are connected with an equal sign.

Key Concepts

- **Algebraic Notation**

Operation	Notation	Say:	The result is…
Addition	$a + b$	a plus b	the sum of a and b
Multiplication	$a \cdot b, (a)(b), (a)b, a(b)$	a times b	The product of a and b
Subtraction	$a - b$	a minus b	the difference of a and b
Division	$a \div b, a/b, \frac{a}{b}, b\overline{)a}$	a divided by b	The quotient of a and b

Understanding Algebraic Notation

- **Algebraic inequalitysymbols**

Algebraic Notation	Say
$a = b$	a is equal to b
$a \neq b$	a is not equal to b
$a < b$	a is less than b
$a > b$	a is greater than b
$a \leq b$	a is less than or equal to b
$a \geq b$	a is greater than or equal to b

Understanding Algebraic Notation

- **Equality Symbol**
  - $a = b$ is read as a is equal to b
  - The symbol $=$ is called the equal sign.
- **Order of Operations** When simplifying mathematical expressions perform the operations in the following order:
  - Parentheses and other Grouping Symbols: Simplify all expressions inside the parentheses or other grouping symbols, working on the innermost parentheses first.
  - Exponents: Simplify all expressions with exponents.
  - Multiplication and Division: Perform all multiplication and division in order from left to right. These operations have equal priority.
  - Addition and Subtraction: Perform all addition and subtraction in order from left to right. These operations have equal priority.
- **Combine like terms.**

  Identify like terms.
  Rearrange the expression so like terms are together.
  Add the coefficients of the like terms

Glossary

Term
: A term is a constant or the product of a constant and one or more variables.

Coefficient

The constant that multiplies the variable(s) in a term is called the coefficient.

Like Terms

Terms that are either constants or have the same variables with the same exponents are like terms.

Evaluate

To evaluate an algebraic expression means to find the value of the expression when the variable is replaced by a given number.

Equations: Lessons 1.A - 1.C
By the end of this section, you will be able to:

- Solve equations using the Subtraction and Addition Properties of Equality
- Solve equations that need to be simplified
- Solve equations with Fraction and decimal cooefficients
- Be familiar with Data and its representations

Note:
In case you missed something in class, or just want to review a specific topic covered in this Module, here is a list of topics covered:

1. Solve Equations Using the Subtraction and Addition [link]
2. Solve Equations Using the Division and Multiplication [link]
3. Solve Equations with Fraction Coefficients [link]
4. Solve Equations with Decimal Coefficients [link]
5. Solve Equations Using a General Strategy [link]
6. Solve Equations That Need to Be Simplified [link]
7. Key Concepts [link]

Solve Equations Using the Subtraction and Addition Properties of Equality [footnote]

Section material derived from Openstax Prealgebra: The Language of Algebra-Solving Equations Using the Subtraction and Addition Properties of Equality

We began our work solving equations in previous chapters. It has been a while since we have seen an equation, so we will review some of the key concepts before we go any further.

We said that solving an equation is like discovering the answer to a puzzle. The purpose in solving an equation is to find the value or values of the variable that make each side of the equation the same. Any value of the variable that makes the equation true is called a solution to the equation. It is the answer to the puzzle.

Note:
A **solution of an equation** is a value of a variable that makes a true statement when substituted into the equation.

In the earlier sections, we listed the steps to determine if a value is a solution. We restate them here.

Note:

Substitute the number for the variable in the equation.
Simplify the expressions on both sides of the equation.
Determine whether the resulting equation is true.

- If it is true, the number is a solution.
- If it is not true, the number is not a solution.

Example:
Determining Whether a Number is a Solution to an Equation
Exercise:

Problem: Determine whether $y = \frac{3}{4}$ is a solution for $4y + 3 = 8y$.

Solution:
Solution

	$4y+3=8y$
Substitute $\frac{3}{4}$ for *y*.	$4\left(\frac{3}{4}\right)+3\stackrel{?}{=}8\left(\frac{3}{4}\right)$
Multiply.	$3+3\stackrel{?}{=}6$
Add.	$6=6\checkmark$

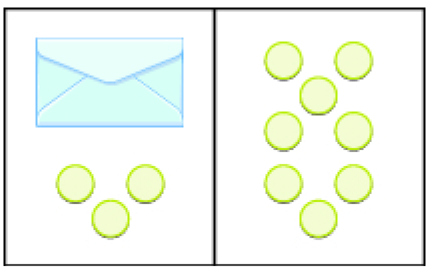
Since $y=\frac{3}{4}$ results in a true equation, $\frac{3}{4}$ is a solution to the equation $4y+3=8y$.

We introduced the Subtraction and Addition Properties of Equality in Solving Equations Using the Subtraction and Addition Properties of Equality. In that section, we modeled how these properties work and then applied them to solving equations with whole numbers. We used these properties again each time we introduced a new system of numbers. Let’s review those properties here.

Note:
Subtraction Property of Equality
For all real numbers a, b, and c, if $a = b$, then $a - c = b - c$.
Addition Property of Equality
For all real numbers a, b, and c, if $a = b$, then $a + c = b + c$.

When you add or subtract the same quantity from both sides of an equation, you still have equality.

We introduced the Subtraction Property of Equality earlier by modeling equations with envelopes and counters. [link] models the equation $x + 3 = 8$.



The goal is to isolate the variable on one side of the equation. So we 'took away' 3 from both sides of the equation and found the solution $x = 5$.

Some people picture a balance scale, as in [link], when they solve equations.

1 mass on each side = balanced

2 masses on each side = balanced

1 mass on one side and 2 masses on the other = unbalanced

The quantities on both sides of the equal sign in an equation are equal, or balanced. Just as with the balance scale, whatever you do to one side of the equation you must also do to the other to keep it balanced.

Let’s review how to use Subtraction and Addition Properties of Equality to solve equations. We need to isolate the variable on one side of the equation. And we check our solutions by substituting the value into the equation to make sure we have a true statement.

Example:
Solving for a Variable
Exercise:

Problem: Solve: $x + 11 = -3$.

Solution:
Solution

To isolate x, we undo the addition of 11 by using the Subtraction Property of Equality.

		$x - 11 = -3$
Subtract 11 from each side to "undo" the addition.		$x + 11 - 11 = -3 - 11$
Simplify.		$x = -14$
Check:	$x - 11 = -3$	
Substitute $x = -14$.	$-14 + 11 \stackrel{?}{=} -3$	
	$-3 = -3$ ✓	

Since $x = -14$ makes $x + 11 = -3$ a true statement, we know that it is a solution to the equation.

In the original equation in the previous example, 11 was added to the x, so we subtracted 11 to ‘undo’ the addition. In the next example, we will need to ‘undo’ subtraction by using the Addition Property of Equality.

Example:
Solving for a Variable
Exercise:

Problem: Solve: $m - 4 = -5$.

Solution:
Solution

		$m + 4 = -5$
Add 4 to each side to "undo" the subtraction.		$m + 4 - 4 = -5 - 4$
Simplify.		$m = -1$
Check:	$m + 4 = -5$	

Substitute $m=-1$.	$-1+4\stackrel{?}{=}-5$	
	$-5=-5$ ✓	
		The solution to $m-4=-5$ is $m=-1$.

Now let's review solving equations with fractions.

Example:
Solving for a Variable with Fractions
Exercise:

Problem: Solve: $n-\frac{3}{8}=\frac{1}{2}$.

Solution:
Solution

	$n-\frac{3}{8}=\frac{1}{2}$
Use the Addition Property of Equality.	$n-\frac{3}{8}+\frac{3}{8}=\frac{1}{2}+\frac{3}{8}$

Find the LCD to add the fractions on the right.		$n - \frac{3}{8} + \frac{3}{8} = \frac{4}{8} + \frac{3}{8}$
Simplify		$n = \frac{7}{8}$
Check:	$n - \frac{3}{8} = \frac{1}{2}$	
Substitute $n = \frac{7}{8}$.	$\frac{7}{8} - \frac{3}{8} \stackrel{?}{=} \frac{1}{2}$	
Subtract.	$\frac{4}{8} \stackrel{?}{=} \frac{1}{2}$	
Simplify.	$\frac{1}{2} = \frac{1}{2}$ ✓	
The solution checks.		

In Solve Equations with Decimals, we solved equations that contained decimals. We'll review this next.

Example:
Solving for a Variable with Decimals
Exercise:

Problem: Solve $a-3.7=4.3$.

Solution:
Solution

		$a-3.7=4.3$
Use the Addition Property of Equality.		$a-3.7+3.7=4.3+3.7$
Add.		$a=8$
Check:	$a-3.7=4.3$	
Substitute $a=8$.	$8-3.7\stackrel{?}{=}4.3$	
Simplify.	$4.3=4.3\checkmark$	
The solution checks.		

Solve Equations Using the Division and Multiplication Properties of Equality [footnote]

Section material derived from Openstax Elementary Algebra: Solving Linear Equations and Inequalities-Solve Equations using the Division and Multiplication Properties of Equality

We introduced the Multiplication and Division Properties of Equality in Solve Equations Using Integers; The Division Property of Equality and Solve Equations with Fractions. We modeled how these properties worked using envelopes and counters and then applied them to solving equations (See Solve Equations Using Integers; The Division Property of Equality). We restate them again here as we prepare to use these properties again.

Note:
Division Property of Equality: For all real numbers a, b, c, and $c \neq 0$, if $a = b$, then $\frac{a}{c} = \frac{b}{c}$.
Multiplication Property of Equality: For all real numbers a, b, c, if $a = b$, then $ac = bc$.

When you divide or multiply both sides of an equation by the same quantity, you still have equality.

Let's review how these properties of equality can be applied in order to solve equations. Remember, the goal is to 'undo' the operation on the variable. In the example below the variable is multiplied by 4, so we will divide both sides by 4 to 'undo' the multiplication.

Example:
Solving for a Variable with Division
Exercise:

Problem: Solve: $4x = -28$.

Solution:
Solution

We use the Division Property of Equality to divide both sides by 4.

	$4x = -28$
Divide both sides by 4 to undo the multiplication.	$\frac{4x}{4} = \frac{-28}{4}$
Simplify.	$x = -7$
Check your answer. Let $x = -7$.	
$4x = -28$	
$4(-7) \stackrel{?}{=} -28$	

$-28 = -28$ ✓	

Since this is a true statement, $x = -7$ is a solution to $4x = -28$.

In the previous example, to 'undo' multiplication, we divided. How do you think we 'undo' division?

Example:
Solving for a Variable with Division
Exercise:

Problem: Solve: $\frac{a}{-7} = -42$.

Solution:
Solution

Here a is divided by -7. We can multiply both sides by -7 to isolate a.

	$\frac{a}{-7} = -42$
Multiply both sides by -7.	$-7\left(\frac{a}{-7}\right) = -7(-42)$

	$\frac{-7a}{-7} = 294$
Simplify.	$a = 294$
Check your answer. Let $a = 294$.	
$\frac{a}{-7} = -42$	
$\frac{294}{-7} \stackrel{?}{=} -42$	
$-42 = -42$ ✓	

Example:
Solving for a Variable with Division
Exercise:

Problem: Solve: $-r = 2$.

Solution:
Solution

Remember $-r$ is equivalent to $-1r$.

		$-r = 2$
Rewrite $-r$ as $-1r$.		$-1r = 2$
Divide both sides by -1.		$\frac{-1r}{-1} = \frac{2}{-1}$
		$r = -2$
Check.	$-r = 2$	
Substitute $r = -2$	$-(-2) \stackrel{?}{=} 2$	
Simplify.	$2 = 2\checkmark$	

In Solve Equations with Fractions, we saw that there are two other ways to solve $-r = 2$.

We could multiply both sides by -1.

We could take the opposite of both sides.

Example:
Solving for a Variable with Fractions
Exercise:

Problem: Solve: $\frac{2}{3}x = 18$.

Solution:
Solution

Since the product of a number and its reciprocal is 1, our strategy will be to isolate x by multiplying by the reciprocal of $\frac{2}{3}$.

	$\frac{2}{3}x = 18$
Multiply by the reciprocal of $\frac{2}{3}$.	$\frac{3}{2} \cdot \frac{2}{3}x = \frac{3}{2} \cdot 18$
Reciprocals multiply to one.	$1x = \frac{3}{2} \cdot \frac{18}{1}$
Multiply.	$x = 27$

Check your answer. Let $x = 27$	
$\frac{2}{3}x = 18$	
$\frac{2}{3} \cdot 27 \stackrel{?}{=} 18$	
$18 = 18 \checkmark$	

Notice that we could have divided both sides of the equation $\frac{2}{3}x = 18$ by $\frac{2}{3}$ to isolate x. While this would work, multiplying by the reciprocal requires fewer steps.

Solve Equations with Fraction Coefficients [footnote]

Section material derived from Openstax Prealgebra: Solving Linear Equations-Solve Equations with Fraction or Decimal Coefficients

Let's use the General Strategy for Solving Linear Equations introduced earlier to solve the equation $\frac{1}{8}x + \frac{1}{2} = \frac{1}{4}$.

	$\frac{1}{8}x+\frac{1}{2}=\frac{1}{4}$
To isolate the x term, subtract $\frac{1}{2}$ from both sides.	$\frac{1}{8}x+\frac{1}{2}-\frac{1}{2}=\frac{1}{4}-\frac{1}{2}$
Simplify the left side.	$\frac{1}{8}x=\frac{1}{4}-\frac{1}{2}$
Change the constants to equivalent fractions with the LCD.	$\frac{1}{8}x=\frac{1}{4}-\frac{2}{4}$
Subtract.	$\frac{1}{8}x=-\frac{1}{4}$
Multiply both sides by the reciprocal of $\frac{1}{8}$.	$\frac{8}{1}\cdot\frac{1}{8}x=\frac{8}{1}\left(-\frac{1}{4}\right)$
Simplify.	$x=-2$

This method worked fine, but many students don't feel very confident when they see all those fractions. So we are going to show an alternate method to solve equations with fractions. This alternate method eliminates the fractions.

We will apply the Multiplication Property of Equality and multiply both sides of an equation by the least common denominator of *all* the fractions in the equation. The result of this operation will be a new equation, equivalent to the first, but with no fractions. This process is called *clearing the*

equation of fractions. Let’s solve the same equation again, but this time use the method that clears the fractions.

Example:
Solving for a Variable with Fractions
Exercise:

Problem: Solve: $\frac{1}{8}x + \frac{1}{2} = \frac{1}{4}$.

Solution:
Solution

Find the least common denominator of *all* the fractions in the equation.	$\frac{1}{8}x + \frac{1}{2} = \frac{1}{4} \quad \text{LCD} = 8$
Multiply both sides of the equation by that LCD, 8. This clears the fractions.	$8\left(\frac{1}{8}x + \frac{1}{2}\right) = 8\left(\frac{1}{4}\right)$
Use the Distributive Property.	$8 \cdot \frac{1}{8}x + 8 \cdot \frac{1}{2} = 8 \cdot \frac{1}{4}$
Simplify — and notice, no more fractions!	$x + 4 = 2$
Solve using the General Strategy for Solving Linear Equations.	$x + 4 - 4 = 2 - 4$
Simplify.	

	$x = -2$
Check: Let $x = -2$ $\frac{1}{8}x + \frac{1}{2} = \frac{1}{4}$ $\frac{1}{8}(-2) + \frac{1}{2} \stackrel{?}{=} \frac{1}{4}$ $-\frac{2}{8} + \frac{1}{2} \stackrel{?}{=} \frac{1}{4}$ $-\frac{2}{8} + \frac{4}{8} \stackrel{?}{=} \frac{1}{4}$ $\frac{2}{4} \stackrel{?}{=} \frac{1}{4}$ $\frac{1}{4} = \frac{1}{4}$ ✓	

Notice in [link] that once we cleared the equation of fractions, the equation was like those we solved earlier in this chapter. We changed the problem to one we already knew how to solve! We then used the General Strategy for Solving Linear Equations.

Note:

Find the least common denominator of *all* the fractions in the equation. Multiply both sides of the equation by that LCD. This clears the fractions. Solve using the General Strategy for Solving Linear Equations.

Example:
Solving for a Variable with Fractions
Exercise:

Problem: Solve: $7 = \frac{1}{2}x + \frac{3}{4}x - \frac{2}{3}x.$

Solution:
Solution

We want to clear the fractions by multiplying both sides of the equation by the LCD of all the fractions in the equation.

Find the least common denominator of *all* the fractions in the equation.	$7 = \frac{1}{2}x + \frac{3}{4}x - \frac{2}{3}x \quad \text{LCD} = 12$
Multiply both sides of the equation by 12.	$12(7) = 12 \cdot \frac{1}{2}x + \frac{3}{4}x - \frac{2}{3}x$
Distribute.	$12(7) = 12 \cdot \frac{1}{2}x + 12 \cdot \frac{3}{4}x - 12 \cdot \frac{2}{3}x$
Simplify — and notice, no more fractions!	$84 = 6x + 9x - 8x$
Combine like terms.	$84 = 7x$
Divide by 7.	$\frac{84}{7} = \frac{7x}{7}$

Simplify.	$12 = x$
Check: Let $x = 12$.	
$7 = \frac{1}{2}x + \frac{3}{4}x - \frac{2}{3}x$ $7 \stackrel{?}{=} \frac{1}{2}(12) + \frac{3}{4}(12) - \frac{2}{3}(12)$ $7 \stackrel{?}{=} 6 + 9 - 8$ $7 = 7 \checkmark$	

In the next example, we'll have variables and fractions on both sides of the equation.

Example:
Solving for a Variable with Fractions
Exercise:

Problem: Solve: $x + \frac{1}{3} = \frac{1}{6}x - \frac{1}{2}$.

Solution:
Solution

Find the LCD of all the fractions in the equation.	$x+\frac{1}{3}=\frac{1}{6}x-\frac{1}{2}$, LCD $= 6$
Multiply both sides by the LCD.	$6\left(x+\frac{1}{3}\right)=6\left(\frac{1}{6}x-\frac{1}{2}\right)$
Distribute.	$6\cdot x+6\cdot\frac{1}{3}=6\cdot\frac{1}{6}x-6\cdot\frac{1}{2}$
Simplify — no more fractions!	$6x+2=x-3$
Subtract x from both sides.	$6x-x+2=x-x-3$
Simplify.	$5x+2=-3$
Subtract 2 from both sides.	$5x+2-2=-3-2$
Simplify.	$5x=-5$
Divide by 5.	$\frac{5x}{5}=\frac{-5}{5}$
Simplify.	$x=-1$

Check: Substitute $x = -1$.

$$x + \frac{1}{3} = \frac{1}{6}x - \frac{1}{2}$$
$$(-1) + \frac{1}{3} \stackrel{?}{=} \frac{1}{6}(-1) - \frac{1}{2}$$
$$(-1) + \frac{1}{3} \stackrel{?}{=} -\frac{1}{6} - \frac{1}{2}$$
$$-\frac{3}{3} + \frac{1}{3} \stackrel{?}{=} -\frac{1}{6} - \frac{3}{6}$$
$$-\frac{2}{3} \stackrel{?}{=} -\frac{4}{6}$$
$$-\frac{2}{3} = -\frac{2}{3} \checkmark$$

In [link], we’ll start by using the Distributive Property. This step will clear the fractions right away!

Example:
Solving for a Variable with Fractions
Exercise:

Problem: Solve: $1 = \frac{1}{2}(4x + 2)$.

Solution:
Solution

	$1 = \frac{1}{2}(4x + 2)$
Distribute.	$1 = \frac{1}{2} \cdot 4x + \frac{1}{2} \cdot 2$
Simplify. Now there are no fractions to clear!	$1 = 2x + 1$
Subtract 1 from both sides.	$1 - 1 = 2x + 1 - 1$
Simplify.	$0 = 2x$
Divide by 2.	$\frac{0}{2} = \frac{2x}{2}$
Simplify.	$0 = x$
Check: Let $x = 0$.	

$$1 = \frac{1}{2}(4x + 2)$$

$$1 \stackrel{?}{=} \frac{1}{2}(4(0) + 2)$$

$$1 \stackrel{?}{=} \frac{1}{2}(2)$$

$$1 \stackrel{?}{=} \frac{2}{2}$$

$$1 = 1 \checkmark$$

Many times, there will still be fractions, even after distributing.

Example:
Solving for a Variable with Fractions
Exercise:

Problem: Solve: $\frac{1}{2}(y-5) = \frac{1}{4}(y-1)$.

Solution:
Solution

	$\frac{1}{2}(y-5) = \frac{1}{4}(y-1)$

Distribute.	$\frac{1}{2} \cdot y - \frac{1}{2} \cdot 5 = \frac{1}{4} \cdot y - \frac{1}{4} \cdot 1$
Simplify.	$\frac{1}{2}y - \frac{5}{2} = \frac{1}{4}y - \frac{1}{4}$
Multiply by the LCD, 4.	$4\left(\frac{1}{2}y - \frac{5}{2}\right) = 4\left(\frac{1}{4}y - \frac{1}{4}\right)$
Distribute.	$4 \cdot \frac{1}{2}y - 4 \cdot \frac{5}{2} = 4 \cdot \frac{1}{4}y - 4 \cdot \frac{1}{4}$
Simplify.	$2y - 10 = y - 1$
Collect the y terms to the left.	$2y - 10 - y = y - 1 - y$
Simplify.	$y - 10 = -1$
Collect the constants to the right.	$y - 10 + 10 = -1 + 10$
Simplify.	$y = 9$
Check: Substitute 9 for y.	

$$\frac{1}{2}(y-5)=\frac{1}{4}(y-1)$$
$$\frac{1}{2}(9-5)\stackrel{?}{=}\frac{1}{4}(9-1)$$
$$\frac{1}{2}(4)\stackrel{?}{=}\frac{1}{4}(8)$$
$$2=2\checkmark$$

Solve Equations with Decimal Coefficients [footnote]

Section material derived from Openstax Prealgebra: Solving Linear Equations-Solve Equations with Fraction or Decimal Coefficients

Some equations have decimals in them. This kind of equation will occur when we solve problems dealing with money and percent. But decimals are really another way to represent fractions. For example, $0.3=\frac{3}{10}$ and $0.17=\frac{17}{100}$. So, when we have an equation with decimals, we can use the same process we used to clear fractions—multiply both sides of the equation by the least common denominator.

Example:
Solving for a Varaible with Decimals
Exercise:

Problem: Solve: $0.8x-5=7$.

Solution:
Solution

The only decimal in the equation is 0.8. Since $0.8 = \frac{8}{10}$, the LCD is 10. We can multiply both sides by 10 to clear the decimal.

	$0.8x - 5 = 7$
Multiply both sides by the LCD.	$10(0.8x - 5) = 10(7)$
Distribute.	$10(0.8x) - 10(5) = 10(7)$
Multiply, and notice, no more decimals!	$8x - 50 = 70$
Add 50 to get all constants to the right.	$8x - 50 + 50 = 70 + 50$
Simplify.	$8x = 120$
Divide both sides by 8.	$\frac{8x}{8} = \frac{120}{8}$
Simplify.	$x = 15$
Check: Let $x = 15$.	

$$0.8(15) - 5 \stackrel{?}{=} 7$$ $$12 - 5 \stackrel{?}{=} 7$$ $$7 = 7 \checkmark$$	

Example:
Solving for a Variable with Decimals
Exercise:

Problem: Solve: $0.06x + 0.02 = 0.25x - 1.5$.

Solution:
Solution

Look at the decimals and think of the equivalent fractions.

$$0.06 = \frac{6}{100}, \quad 0.02 = \frac{2}{100}, \quad 0.25 = \frac{25}{100}, \quad 1.5 = 1\frac{5}{10}$$

Notice, the LCD is 100.

By multiplying by the LCD we will clear the decimals.

	$0.06x + 0.02 = 0.25x - 1.5$

Multiply both sides by 100.	$100(0.06x + 0.02) = 100(0.25x - 1.5)$
Distribute.	$100(0.06x) + 100(0.02) = 100(0.25x) - 100(1.5)$
Multiply, and now no more decimals.	$6x + 2 = 25x - 150$
Collect the variables to the right.	$6x - 6x + 2 = 25x - 6x - 150$
Simplify.	$2 = 19x - 150$
Collect the constants to the left.	$2 + 150 = 19x - 150 + 150$
Simplify.	$152 = 19x$
Divide by 19.	$\frac{152}{19} = \frac{19x}{19}$
Simplify.	$8 = x$
Check: Let $x = 8$.	
$0.06(8) + 0.02 = 0.25(8) - 1.5$ $0.48 + 0.02 = 2.00 - 1.5$ $0.50 = 0.50 \checkmark$	

The next example uses an equation that is typical of the ones we will see in the money applications in the next chapter. Notice that we will distribute the decimal first before we clear all decimals in the equation.

Example:
Solving for a Variable with Decimals
Exercise:

Problem: Solve: $0.25x + 0.05(x + 3) = 2.85$.

Solution:
Solution

	$0.25x + 0.05(x + 3) = 2.85$
Distribute first.	$0.25x + 0.05x + 0.15 = 2.85$
Combine like terms.	$0.30x + 0.15 = 2.85$
To clear decimals, multiply by 100.	$100(0.30x + 0.15) = 100(2.85)$

Distribute.	$30x + 15 = 285$
Subtract 15 from both sides.	$30x + 15 - 15 = 285 - 15$
Simplify.	$30x = 270$
Divide by 30.	$\frac{30x}{30} = \frac{270}{30}$
Simplify.	$x = 9$
Check: Let $x = 9$.	
$0.25x + 0.05(x + 3) = 2.85$ $0.25(9) + 0.05(9 + 3) \stackrel{?}{=} 2.85$ $2.25 + 0.05(12) \stackrel{?}{=} 2.85$ $2.25 + 0.60 \stackrel{?}{=} 2.85$ $2.85 = 2.85$ ✓	

Note:
ACCESS ADDITIONAL ONLINE RESOURCES

- Solve an Equation with Fractions with Variable Terms on Both Sides

Solve Equations Using a General Strategy [footnote]

Section material derived from Openstax Prealgebra: Solving Linear Equations-Solve Equations with Variables and Constants on Both Sides

Each of the first few sections of this chapter has dealt with solving one specific form of a linear equation. It's time now to lay out an overall strategy that can be used to solve *any* linear equation. We call this the *general strategy*. Some equations won't require all the steps to solve, but many will. Simplifying each side of the equation as much as possible first makes the rest of the steps easier.

Note:

Simplify each side of the equation as much as possible. Use the Distributive Property to remove any parentheses. Combine like terms.
Collect all the variable terms to one side of the equation. Use the Addition or Subtraction Property of Equality.
Collect all the constant terms to the other side of the equation. Use the Addition or Subtraction Property of Equality.
Make the coefficient of the variable term to equal to 1. Use the Multiplication or Division Property of Equality. State the solution to the equation.
Check the solution. Substitute the solution into the original equation to

make sure the result is a true statement.

Example:
Solving Equations
Exercise:

Problem: Solve: $3(x + 2) = 18$.

Solution:
Solution

	$3(x + 2) = 18$
Simplify each side of the equation as much as possible. Use the Distributive Property.	$3x + 6 = 18$
Collect all variable terms on one side of the equation—all xs are already on the left side.	
Collect constant terms on the other side of the equation. Subtract 6 from each side	$3x + 6 - 6 = 18 - 6$
Simplify.	$3x = 12$
Make the coefficient of the variable term equal to	

1. Divide each side by 3.	$\frac{3x}{3} = \frac{12}{3}$
Simplify.	$x = 4$
Check: Let $x = 4$.	
$$\begin{aligned} 3(x+2) &= 18 \\ 3(4+2) &\stackrel{?}{=} 18 \\ 3(6) &\stackrel{?}{=} 18 \\ 18 &= 18 \checkmark \end{aligned}$$	

Example:
Solving Equations
Exercise:

Problem: Solve: $-(x+5) = 7$.

Solution:
Solution

	$-(x+5) = 7$

Simplify each side of the equation as much as possible by distributing. The only x term is on the left side, so all variable terms are on the left side of the equation.	$-x-5=7$
Add 5 to both sides to get all constant terms on the right side of the equation.	$-x-5+5=7+5$
Simplify.	$-x=12$
Make the coefficient of the variable term equal to 1 by multiplying both sides by -1.	$-1(-x)=-1(12)$
Simplify.	$x=-12$
Check: Let $x=-12$.	
$-(x+5)=7$ $-(-12+5)\stackrel{?}{=}7$ $-(-7)\stackrel{?}{=}7$ $7=7\checkmark$	

Example:
Solving Equations
Exercise:

Problem: Solve: $4(x-2)+5=-3$.

Solution:
Solution

	$4(x-2)+5=-3$
Simplify each side of the equation as much as possible. Distribute.	$4x-8+5=-3$
Combine like terms	$4x-3=-3$
The only x is on the left side, so all variable terms are on one side of the equation.	
Add 3 to both sides to get all constant terms on the other side of the equation.	$4x-3+3=-3+3$
Simplify.	$4x=0$
Make the coefficient of the variable term equal to 1 by dividing both sides by 4.	$\frac{4x}{4}=\frac{0}{4}$

Simplify.	$x = 0$
Check: Let $x = 0$.	
$$\begin{aligned} 4(x-2)+5 &= -3 \\ 4(0-2)+5 &\stackrel{?}{=} -3 \\ 4(-2)+5 &\stackrel{?}{=} -3 \\ -8+5 &\stackrel{?}{=} -3 \\ -3 &= -3 \checkmark \end{aligned}$$	

Example:
Solving Equations
Exercise:

Problem: Solve: $8 - 2(3y + 5) = 0$.

Solution:
Solution

Be careful when distributing the negative.

	$8 - 2(3y + 5) = 0$
Simplify—use the Distributive Property.	

	$8 - 6y - 10 = 0$
Combine like terms.	$-6y - 2 = 0$
Add 2 to both sides to collect constants on the right.	$-6y - 2 + 2 = 0 + 2$
Simplify.	$-6y = 2$
Divide both sides by −6.	$\frac{-6y}{-6} = \frac{2}{-6}$
Simplify.	$y = -\frac{1}{3}$
Check: Let $y = -\frac{1}{3}$.	
$\begin{aligned} 8 - 2(3y + 5) &= 0 \\ 8 - 2\left[3\left(-\frac{1}{3}\right) + 5\right] &= 0 \\ 8 - 2(-1 + 5) &\stackrel{?}{=} 0 \\ 8 - 2(4) &\stackrel{?}{=} 0 \\ 8 - 8 &\stackrel{?}{=} 0 \\ 0 &= 0 \checkmark \end{aligned}$	

Example:
Solving Equations
Exercise:

Problem: Solve: $3(x-2)-5=4(2x+1)+5$.

Solution:
Solution

	$3(x-2)-5=4(2x+1)+5$
Distribute.	$3x-6-5=8x+4+5$
Combine like terms.	$3x-11=8x+9$
Subtract $3x$ to get all the variables on the right since $8>3$.	$3x-3x-11=8x-3x+9$
Simplify.	$-11=5x+9$
Subtract 9 to get the constants on the left.	$-11-9=5x+9-9$
Simplify.	$-20=5x$

Divide by 5.	$\frac{-20}{5} = \frac{5x}{5}$
Simplify.	$-4 = x$
Check: Substitute: $-4 = x$.	
$$\begin{aligned} 3(x-2)-5 &= 4(2x+1)+5 \\ 3(-4-2)-5 &\stackrel{?}{=} 4[2(-4)+1]+5 \\ 3(-6)-5 &\stackrel{?}{=} 4(-8+1)+5 \\ -18-5 &\stackrel{?}{=} 4(-7)+5 \\ -23 &\stackrel{?}{=} -28+5 \\ -23 &= -23 \checkmark \end{aligned}$$	

Example:
Solving Equations
Exercise:

Problem: Solve: $\frac{1}{2}(6x-2) = 5 - x$.

Solution:
Solution

	$\frac{1}{2}(6x-2)=5-x$
Distribute.	$3x-1=5-x$
Add x to get all the variables on the left.	$3x-1+x=5-x+x$
Simplify.	$4x-1=5$
Add 1 to get constants on the right.	$4x-1+1=5+1$
Simplify.	$4x=6$
Divide by 4.	$\frac{4x}{4}=\frac{6}{4}$
Simplify.	$x=\frac{3}{2}$
Check: Let $x=\frac{3}{2}$.	

$$\frac{1}{2}(6x-2)=5-x$$

$$\frac{1}{2}\left(6\cdot\frac{3}{2}-2\right)\stackrel{?}{=}5-\frac{3}{2}$$

$$\frac{1}{2}(9-2)\stackrel{?}{=}\frac{10}{2}-\frac{3}{2}$$

$$\frac{1}{2}(7)\stackrel{?}{=}\frac{7}{2}$$

$$\frac{7}{2}=\frac{7}{2}\checkmark$$

In many applications, we will have to solve equations with decimals. The same general strategy will work for these equations.

Example:
Solving Equations
Exercise:

Problem: Solve: $0.24(100x+5)=0.4(30x+15)$.

Solution:
Solution

	$0.24(100x+5)=0.4(30x+15)$

Distribute.	$24x + 1.2 = 12x + 6$
Subtract $12x$ to get all the xs to the left.	$24x + 1.2 - 12x = 12x + 6 - 12x$
Simplify.	$12x + 1.2 = 6$
Subtract 1.2 to get the constants to the right.	$12x + 1.2 - 1.2 = 6 - 1.2$
Simplify.	$12x = 4.8$
Divide.	$\frac{12x}{12} = \frac{4.8}{12}$
Simplify.	$x = 0.4$
Check: Let $x = 0.4$.	
$\begin{array}{c} 0.24(100x + 5) = 0.4(30x + 15) \\ 0.24(100(0.4) + 5) \stackrel{?}{=} 0.4(30(0.4) + 15) \\ 0.24(40 + 5) \stackrel{?}{=} 0.4(12 + 15) \\ 0.24(45) \stackrel{?}{=} 0.4(27) \\ 10.8 = 10.8 \checkmark \end{array}$	

Note:
ACCESS ADDITIONAL ONLINE RESOURCES

- Solving Multi-Step Equations
- Solve an Equation with Variable Terms on Both Sides
- Solving Multi-Step Equations (L5.4)
- Solve an Equation with Variables and Parentheses on Both Sides

Solve Equations That Need to Be Simplified [footnote]

Section material derived from Openstax Prealgebra: The Language of Algebra-Solving Equations Using the Subtraction and Addition Properties of Equality

In the examples up to this point, we have been able to isolate the variable with just one operation. Many of the equations we encounter in algebra will take more steps to solve. Usually, we will need to simplify one or both sides of an equation before using the Subtraction or Addition Properties of Equality. You should always simplify as much as possible before trying to isolate the variable.

Example:
Simplify and Solve
Exercise:

Problem: Solve: $3x - 7 - 2x - 4 = 1$.

Solution:
Solution

The left side of the equation has an expression that we should simplify before trying to isolate the variable.

	$3x - 7 - 2x - 4 = 1$
Rearrange the terms, using the Commutative Property of Addition.	$3x - 2x - 7 - 4 = 1$
Combine like terms.	$x - 11 = 1$
Add 11 to both sides to isolate x.	$x - 11 + 11 = 1 + 11$
Simplify.	$x = 12$
Check. Substitute $x = 12$ into the original equation. $\begin{aligned} 3x - 7 - 2x - 4 &= 1 \\ 3(12) - 7 - 2(12) - 4 &= 1 \\ 36 - 7 - 24 - 4 &= 1 \\ 29 - 24 - 4 &= 1 \\ 5 - 4 &= 1 \\ 1 &= 1 \checkmark \end{aligned}$	

The solution checks.

Example:
Simplify and Solve
Exercise:

Problem: Solve: $3(n-4)-2n=-3$.

Solution:
Solution

The left side of the equation has an expression that we should simplify.

	$3(n-4)-2n=-3$
Distribute on the left.	$3n-12-2n=-3$
Use the Commutative Property to rearrange terms.	$3n-2n-12=-3$
Combine like terms.	$n-12=-3$
Isolate n using the Addition Property of Equality.	$n-12+12=-3+12$
Simplify.	$n=9$
Check. Substitute $n=9$ into the original	

equation.

$$
\begin{aligned}
3(n-4)-2n &= -3 \\
3(9-4)-2\cdot 9 &= -3 \\
3(5)-18 &= -3 \\
15-18 &= -3 \\
-3 &= -3\ \checkmark
\end{aligned}
$$

The solution checks.

Example:
Simplify and Solve
Exercise:

Problem: Solve: $2(3k-1)-5k=-2-7$.

Solution:
Solution

Both sides of the equation have expressions that we should simplify before we isolate the variable.

	$2(3k-1)-5k=-2-7$
Distribute on the left, subtract on the right.	$6k-2-5k=-9$

Use the Commutative Property of Addition.	$6k - 5k - 2 = -9$
Combine like terms.	$k - 2 = -9$
Undo subtraction by using the Addition Property of Equality.	$k - 2 + 2 = -9 + 2$
Simplify.	$k = -7$
Check. Let $k = -7$. $2(3k - 1) - 5k = -2 - 7$ $2(3(-7) - 1) - 5(-7) = -2 - 7$ $2(-21 - 1) - 5(-7) = -9$ $2(-22) + 35 = -9$ $-44 + 35 = -9$ $-9 = -9$ ✓ The solution checks.	

Key Concepts

- **Determine whether a number is a solution to an equation.**

  Substitute the number for the variable in the equation.
  Simplify the expressions on both sides of the equation.

Determine whether the resulting equation is true.

If it is true, the number is a solution.
If it is not true, the number is not a solution.

- **Subtraction and Addition Properties of Equality**
  - **Subtraction Property of Equality**
    For all real numbers a, b, and c,
    if $a = b$ then $a - c = b - c$.
  - **Addition Property of Equality**
    For all real numbers a, b, and c,
    if $a = b$ then $a + c = b + c$.

- **Translate a word sentence to an algebraic equation.**

  Locate the “equals” word(s). Translate to an equal sign.
  Translate the words to the left of the “equals” word(s) into an algebraic expression.
  Translate the words to the right of the “equals” word(s) into an algebraic expression.

- **Problem-solving strategy**

  Read the problem. Make sure you understand all the words and ideas.
  Identify what you are looking for.
  Name what you are looking for. Choose a variable to represent that quantity.
  Translate into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebra equation.
  Solve the equation using good algebra techniques.
  Check the answer in the problem and make sure it makes sense.
  Answer the question with a complete sentence.

- **Solve an equation with variables and constants on both sides**

  Choose one side to be the variable side and then the other will be the constant side.

Collect the variable terms to the variable side, using the Addition or Subtraction Property of Equality.
Collect the constants to the other side, using the Addition or Subtraction Property of Equality.
Make the coefficient of the variable 1, using the Multiplication or Division Property of Equality.
Check the solution by substituting into the original equation.

- **General strategy for solving linear equations**

  Simplify each side of the equation as much as possible. Use the Distributive Property to remove any parentheses. Combine like terms.
  Collect all the variable terms to one side of the equation. Use the Addition or Subtraction Property of Equality.
  Collect all the constant terms to the other side of the equation. Use the Addition or Subtraction Property of Equality.
  Make the coefficient of the variable term to equal to 1. Use the Multiplication or Division Property of Equality. State the solution to the equation.
  Check the solution. Substitute the solution into the original equation to make sure the result is a true statement.

- **Division and Multiplication Properties of Equality**

  - **Division Property of Equality:** For all real numbers a, b, c, and $c \neq 0$, if $a = b$, then $ac = bc$.
  - **Multiplication Property of Equality:** For all real numbers a, b, c, if $a = b$, then $ac = bc$.

- **Solve equations with fraction coefficients by clearing the fractions.**

  Find the least common denominator of*all*the fractions in the equation.
  Multiply both sides of the equation by that LCD. This clears the fractions.
  Solve using the General Strategy for Solving Linear Equations.

Glossary

Solution of an Equation

A solution to an equation is a value of a variable that makes a true statement when substituted into the equation. The process of finding the solution to an equation is called solving the equation.

Election Maths: Lessons 1.C - 1.D

Note:
In case you missed something in class, or just want to review a specific topic covered in this Module, here is a list of topics covered:

1. Simple Majority Elections [link]
2. Instant Runoff Elections [link]
3. Borda Count Method Elections [link]
4. Key Concepts [link]

Simple Majority Elections [footnote]

This material was created by Amanda Towry .

Elections are ways of helping large groups of people who have different ideas in electing persons, ideas, activities, and more. Elections are used to elect our political representatives and can also be used to decide what food options will be made for a group party or gathering.

Most elections that occur in your everyday life are simple **majority** elections. Majority elections require a number of votes over 50%, or a majority of the votes cast.This amount can be easily found by simply calculating 50% of the total votes cast in an election.

Example:
Simple Majority Election
Exercise:

Problem:

Janna, Ryan, and Eric are running against each other for the position of class president. Janna is trying to figure out how many votes she would need to win the election. According to her school's records, there are 356 students in Janna's class that she is running to represent. If every student votes in the election, how many of the students must vote for Janna to win the election by a simple majority?

Solution:

If the total number of votes cast is 356, then we simply need to calculate $50\ \%$,

Setting up an equation with the portion of the votes that are unknown is represented as a variable.

$356*0.50 = \mathrm{x}$

Solving for the unknown variable x, gives a value of 178.

However, this is not the actual amount of votes that Janna needs to win the election. Janna needs **more** than 50%. This means the total necessary votes needed must be rounded up by one, giving a total of 179 votes needed for Janna to win the election by simple majority.

Intant Run-Off Elections [footnote]

This material was created by Amanda Towry .

Sometimes a simple majority election cannot be used; usually because an individual does not meet the required greater than 50% of the vote required.

In the case where this happens, an **instant-runoff** election, or elimination method, might be held. An instant runoff is an election where voting choices are ranked by preference and eliminations are made when no one candidate receives a simple majority.

Example:
Instant Run-Off Election
The student council election is run by an instant runoff of a preference schedule. Students are asked to rank their preference for each student running for office as either first, second, or third choice. These preference schedules are then combined and presented in a total votes preference schedule to determine the winner of the student body election.

	71 voters	**97 voters**	**49 voters**	**78 voters**	**61 voters**
1^{st}	Eric	Ryan	Ryan	Janna	Janna
2^{nd}	Janna	Eric	Janna	Eric	Ryan
3^{rd}	Ryan	Janna	Eric	Ryan	Eric

Preference schedule for the student governemnt election.

Adding up all of the first place votes for each of the candidates gives the following results:

- **Eric:** $97 \text{ votes} + 49 \text{ votes} = 146 \text{ votes}$
- **Janna:** $78 \text{ votes} + 61 \text{ votes} = 139 \text{ votes}$
- **Ryan:** 146 votes

It is clear that no one candidate received a simple majority of the vote in this election. In this case, the candidate that received the least amount of votes is eliminated: Ryan. Once Ryan is removed from the ballot, all remaining candidates are moved up in the ranking and first-place votes are recalculated.

	71 voters	**97 voters**	**49 voters**	**78 voters**	**61 voters**
1^{st}	Eric			Janna	Janna
2^{nd}	Janna	Eric	Janna	Eric	
3^{rd}		Janna	Eric		Eric

Preference schedule for the student governemnt election that has had the last place candidate removed from the

ballot for an instant runoff decision.

	71 voters	97 voters	49 voters	78 voters	61 voters
1^{st}	Eric	Eric	Janna	Janna	Janna
2^{nd}	Janna	Janna	Eric	Eric	Eric

Preference schedule for the student governemnt election that has had the next in line behind the eliminated candidate moved up before recalcualting the number of first place votes.

Now, recalculating the total votes shows that:

- **Eric:** $71 \text{ votes} + 97 \text{ votes} = 168 \text{ votes}$
- **Janna:** $49 \text{ votes} + 78 \text{ votes} + 61 \text{ votes} = 188 \text{ votes}$

After the instant runoff, Janna has a total of 188 votes and can now be declared the winner of the election because 188 is more than 50% of the total votes in the class.

Borda Count Method of Election [footnote]

This material was created by Amanda Towry .

In the instant runoff election, a preference schedule was constructed in order to determine the winner of the student body election. Sometimes a preference schedule can be used to determine the winner of an election using points, in a method called a **Borda Count method**.

The Borda Count Method involves awarding points to candidates based on their positions in the preference schedules collected. The lowest position, or last place, always receives the least amount of points and the highest position, or first place, always receives the most points.

The total amount of points that the candidate receives is the total of all the points awarded for all positions in the preference schedule, instead of just in the first place. The person with the most amount of points wins that election.

Example:
Borda Count Election
The student council election is run by the Borda Count method from a preference schedule distributed to the student body. Students are asked to rank their preference for each student running for office as either first, second, or third choice. These preference schedules are then combined and presented in a total votes preference schedule to determine the winner of the student body election.
We start with the initial preference schedule that was completed. Each of the places gets a set amount of points. First place gets 3 points, second place gets two points, and third place gets one point. This is because the last place always gets the least amount of points and the first place vote always gets the most amount of points; with the point value ascending upwards between last to first place.

	71 voters	97 voters	49 voters	78 voters	61 voters
$1^{st}(3\text{ points})$	Eric	Ryan	Ryan	Janna	Janna
$2^{nd}(2\text{ points})$	Janna	Eric	Janna	Eric	Ryan
$3^{rd}(1\text{ point})$	Ryan	Janna	Eric	Ryan	Eric

Preference schedule for the student governemnt election that has the point values of each preference position listed for calculating points awarded to each candidate.

To calculate the number of points each vote in the preference gets, we multiply the place value of the row with the place value of the column each vote occurs in. So In the first column, Eric got first place and his points for that vote are calculated by 3 points (for being in first place), times 71 votes because that column had 71 voters assigned to it.

Equation:

$$(3\ )(\ 71) = 213$$

This process is then repeated for each cell in the preference schedule.

	71 voters	97 voters	49 voters
$1^{st}(3\text{ points})$	Eric $(3{*}71 = 213\text{ points})$	Ryan$(3{*}97 = 291\text{ points})$	Ryan$(3{*}49 = 147\text{ points})$
$2^{nd}(2\text{ points})$	Janna$(2{*}71 = 142\text{ points})$	Eric(194 points)	Janna(98 points)
$2^{nd}(1\text{ point})$	Ryan$(1{*}71 = 71\text{ points})$	Janna(97 points)	Eric(49 points)

Preference schedule for the student governemnt election that has the point values of each preference position calcula

Adding up all of the points each candidate earns for each place vote gives the following totals:

- **Eric:** 213 points $+ 194$ points $+ 49$ points $+ 156$ points $+ 61$ points $= 673$ points
- **Janna:** 142 points $+ 97$ points $+ 98$ points $+ 234$ points $+ 183$ points $= 754$ points
- **Ryan:** 71 points $+ 291$ points $+ 147$ points $+ 78$ points $+ 122$ points $= 709$ points

As can be seen from the totals above, Janna received the highest number of points in this election and thus would be caonsidered the winner in the Borda Count Method election.

Note: Material from this section was developed and edited by Amanda Towry. Material Concept was inspired by the Dana Center materials.

Key Concepts

- **Winners of elections can be conducted in three common ways**

- Simple majority
- Instant run-off
- Borda count method

- **Simple majority elections require at least 51% of the votes cast in the election.**
- **Instant run-off elections are similar to simple majority elections but require eleminating a candidate/option.**
- **Borda count method elections require calculating and adding up points calculating by place vote and number of votes earned for that cadidate/option.**

Glossary

Simple Majority Election
: An election in which a candidate is required to achieve at least 51% of the votes cast in order to be considered the winner.

Instant Run-off Election
: An election in which a candidate/option is eliminated from a ballot or preference schedule in order to achieve a simple majority possibility vote collection for one of the remaining candidates/options.

Borda Count Method
: An election method in which candidates/options in a preference schedule are assigned point values based off ranking position and number of votes cast for that ranking line up.

Measures of Central Tendency: Lessons 2.A - 2.C

Note:
In case you missed something in class, or just want to review a specific topic covered in this Module, here is a list of topics covered:

1. Types of Data and Their Representations [link]
2. Mean, Median, Mode [link]
3. 5-number Summary [link]
4. Box Plots [link]
5. Key Concepts [link]

Types of Data and Their Representations [footnote]

Section material derived from Openstax Introductory Statistics: Sampling and Data-Data Sampling and Variation in Data and Sampling

Data may come from a population or from a sample. Lowercase letters like x or y generally are used to represent data values. Most data can be put into the following categories:

- Qualitative
- Quantitative

Qualitative data are the result of categorizing or describing attributes of a population. Qualitative data are also often called **categorical data**. Hair color, blood type, ethnic group, the car a person drives, and the street a person lives on are examples of qualitative data. Qualitative data are generally described by words or letters. For instance, hair color might be black, dark brown, light brown, blonde, gray, or red. Blood type might be AB+, O-, or B+. Researchers often prefer to use quantitative data over qualitative data because it lends itself more easily to mathematical analysis.

For example, it does not make sense to find an average hair color or blood type.

Quantitative data are always numbers. Quantitative data are the result of **counting** or **measuring** attributes of a population. Amount of money, pulse rate, weight, number of people living in your town, and number of students who take statistics are examples of quantitative data. Quantitative data may be either **discrete** or **continuous**.

All data that are the result of counting are called **quantitative discrete data**. These data take on only certain numerical values. If you count the number of phone calls you receive for each day of the week, you might get values such as zero, one, two, or three.

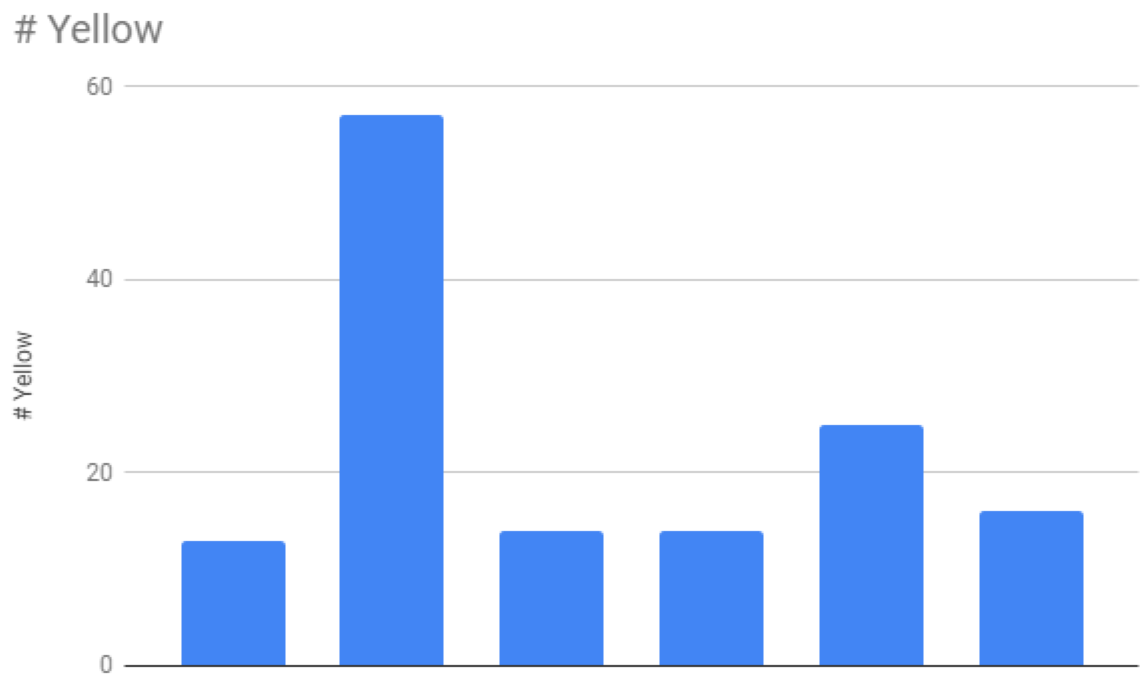
Data that are not only made up of counting numbers, but that may include fractions, decimals, or irrational numbers, are called **quantitative continuous data**. Continuous data are often the results of measurements like lengths, weights, or times. A list of the lengths in minutes for all the phone calls that you make in a week, with numbers like 2.4, 7.5, or 11.0, would be quantitative continuous data.

Tables are a good way of organizing and displaying data. But graphs can be even more helpful in understanding the data. There are no strict rules concerning which graphs to use.

Two graphs that are used to display qualitative data are pie charts and bar graphs.

In a **pie chart**, categories of data are represented by wedges in a circle and are proportional in size to the percent of individuals in each category.

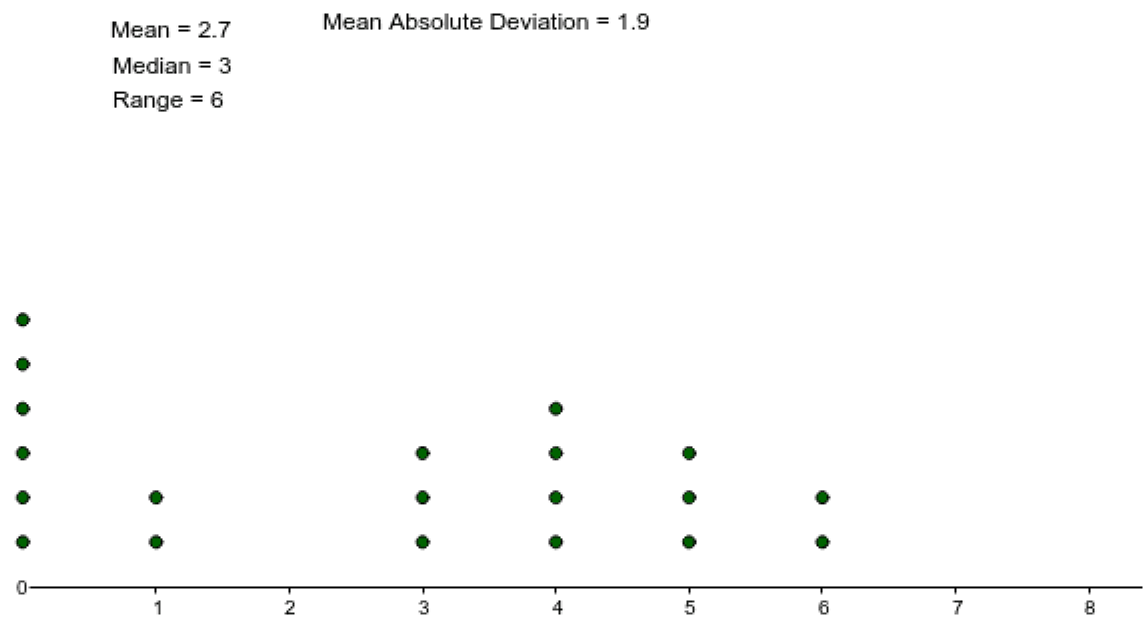
In a **bar graph**, the length of the bar for each category is proportional to the number or percent of individuals in each category. Bars may be vertical or horizontal.



Credit: Created by Amanda Towry by use of Microsoft Excel

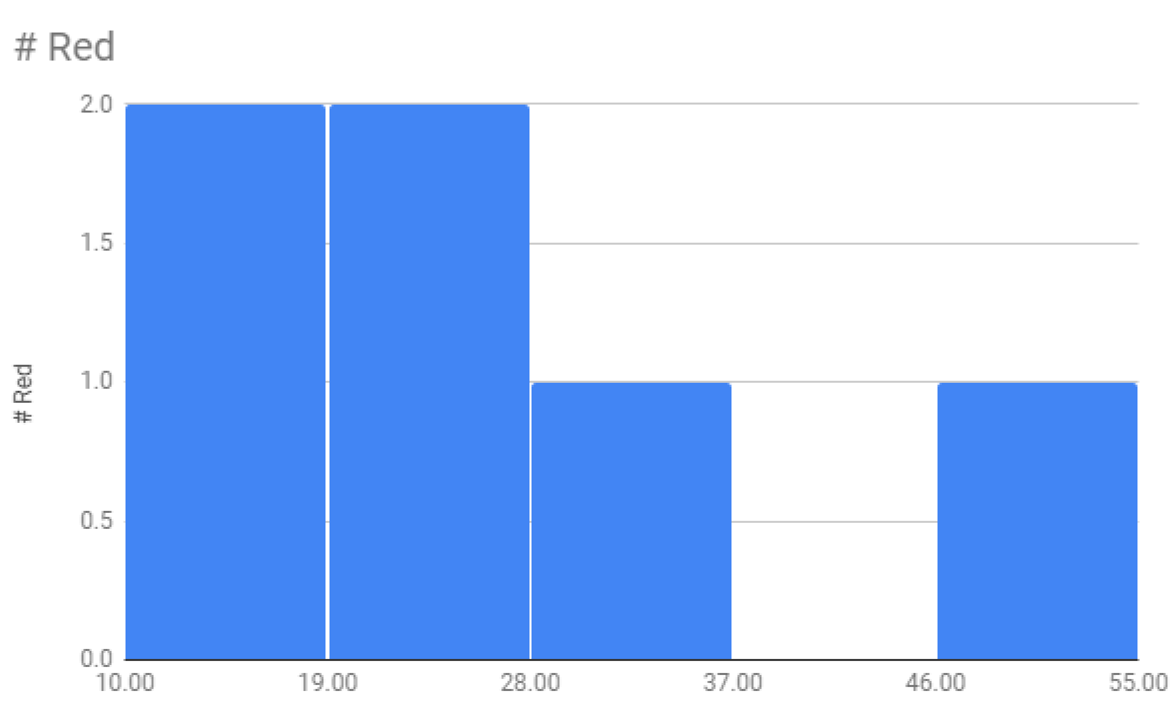
Three graphs that are used to display quantitative data are dot plots, histograms, and box plots.

In a **dot plot**, individual dots are used to represent individual data values on a number line of values.



Credit: Created by Amanda Towry by use of Geogebra Dot Plot Tool found at https://www.geogebra.org/m/HD6hhQ75

In a **histogram**, data is represented similar to a bar graph in structure. However, there is usually no distance or space between the groups of data, called bins, which contain a range of data values instead of a single unique data value/description.



Credit: Created by Amanda Towry by use of Microsoft Excel

In a **box plot** (also called a box and whisker plot), **a five-number summary** is used to construct a visual display of the spread of the data. The five-number summary includes the minimum, the first quartile, the median, the third quartile, and the maximum.

It is a good idea to look at a variety of graphs to see which is the most helpful in displaying the data. We might make different choices of what we think is the "best" graph depending on the data and the context. Our choice also depends on what we are using the data for.

Example:
Types of Data
The data are the number of books students carry in their backpacks. You sample five students. Two students carry three books, one student carries four books, one student carries two books, and one student carries one

book. The numbers of books (three, four, two, and one) are the quantitative discrete data.

Example:
Types of Data
The data are the weights of backpacks with books in them. You sample the same five students. The weights (in pounds) of their backpacks are 6.2, 7, 6.8, 9.1, 4.3. Notice that backpacks carrying three books can have different weights. Weights are quantitative continuous data.

Example:
Types of Data
You go to the supermarket and purchase three cans of soup (19 ounces tomato bisque, 14.1 ounces lentil, and 19 ounces Italian wedding), two packages of nuts (walnuts and peanuts), four different kinds of vegetable (broccoli, cauliflower, spinach, and carrots), and two desserts (16 ounces pistachio ice cream and 32 ounces chocolate chip cookies).
Exercise:

Problem:

Name data sets that are quantitative discrete, quantitative continuous, and qualitative.

Solution:

One Possible Solution:

- The three cans of soup, two packages of nuts, four kinds of vegetables and two desserts are quantitative discrete data because you count them.
- The weights of the soups (19 ounces, 14.1 ounces, 19 ounces) are quantitative continuous data because you measure weights as precisely as possible.

- Types of soups, nuts, vegetables and desserts are qualitative data because they are categorical.

Try to identify additional data sets in this example.

Example:
Types of Data
The data are the colors of backpacks. Again, you sample the same five students. One student has a red backpack, two students have black backpacks, one student has a green backpack, and one student has a gray backpack. The colors red, black, black, green, and gray are qualitative data.

The Mean, Median, and Mode [footnote]

Section material derived from Openstax Prealgebra: Decimals-Averages and Probability

The mean is often called the arithmetic average. It is computed by dividing the sum of the values by the number of values. Students want to know the mean of their test scores. Climatologists report that the mean temperature has, or has not, changed. City planners are interested in the mean household size.

The words “mean” and “average” are often used interchangeably. The substitution of one word for the other is common practice. The technical term is “arithmetic mean” and “average” is technically a center location. However, in practice among non-statisticians, “average" is commonly accepted for “arithmetic mean.”
Equation:

$$\text{mean} = \frac{\text{sum of values in data set}}{n}$$

Suppose Ethan’s first three test scores were $85, 88,$ and 94. To find the mean score, he would add them and divide by 3.
Equation:

$$\frac{85+88+94}{3}$$
$$\frac{267}{3}$$
$$89$$

His mean test score is 89 points.

Note:
The **mean** of a set of n numbers is the arithmetic average of the numbers.

Write the formula for the mean **Equation:**

$$\text{mean} = \frac{\text{sum of values in data set}}{n}$$

Find the sum of all the values in the set. Write the sum in the numerator.
Count the number, n, of values in the set. Write this number in the denominator.
Simplify the fraction.
Check to see that the mean is reasonable. It should be greater than the least number and less than the greatest number in the set.

Example:
Finding the Mean
Exercise:

Problem: Find the mean of the numbers $8, 12, 15, 9,$ and 6.

Solution:
Solution

Write the formula for the mean:	$\text{mean} = \frac{\text{sum of all the numbers}}{n}$
Write the sum of the numbers in the numerator.	$\text{mean} = \frac{8+12+15+9+6}{n}$
Count how many numbers are in the set. There are 5 numbers in the set, so $n = 5$.	$\text{mean} = \frac{8+12+15+9+6}{5}$
Add the numbers in the numerator.	$\text{mean} = \frac{50}{5}$
Then divide.	$\text{mean} = 10$
Check to see that the mean is 'typical': 10 is neither less than 6 nor greater than 15.	The mean is 10.

It is customary to report the mean to one more decimal place than the original numbers. For example, if the numbers represent money, then it will make sense to report the mean in dollars and cents.

The "center" of a data set is also a way of describing location. The two most widely used measures of the "center" of the data are the **mean** (average)

and the **median**. To calculate the **mean weight** of 50 people, add the 50 weights together and divide by 50. To find the **median weight** of the 50 people, order the data and find the number that splits the data into two equal parts. The median is generally a better measure of the center when there are extreme values or outliers because it is not affected by the precise numerical values of the outliers. The mean is the most common measure of the center.

Note:
The **median** of a set of data values is the middle value.

- Half the data values are less than or equal to the median.
- Half the data values are greater than or equal to the median.

Note:

List the numbers from smallest to largest.
Count how many numbers are in the set. Call this n.
Is n odd or even?

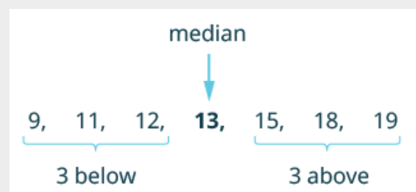
- If n is an odd number, the median is the middle value.
- If n is an even number, the median is the mean of the two middle values.

Example:
Finding the Median
Exercise:

Problem: Find the median of $12, 13, 19, 9, 11, 15,$ and 18.

Solution:

Solution

List the numbers in order from smallest to largest.	9, 11, 12, 13, 15, 18, 19
Count how many numbers are in the set. Call this n.	$n = 7$
Is n odd or even?	odd
The median is the middle value.	
The middle is the number in the 4th position.	So the median of the data is 13.

Example:
Finding the Median
Exercise:

Problem:

Kristen received the following scores on her weekly math quizzes:

$83, 79, 85, 86, 92, 100, 76, 90, 88, \text{and } 64.$ Find her median score.

Solution:
Solution

Find the median of 83, 79, 85, 86, 92, 100, 76, 90, 88, and 64.	
List the numbers in order from smallest to largest.	64, 76, 79, 83, 85, 86, 88, 90, 92, 100
Count the number of data values in the set. Call this n.	$n = 10$
Is n odd or even?	even
The median is the mean of the two middle values, the 5th and 6th numbers.	$\underbrace{64, 76, 79, 83, \mathbf{85},}_{\text{5 numbers}} \quad \underbrace{\mathbf{86,}\ 88, 90, 92, 100}_{\text{5 numbers}}$
Find the mean of 85 and 86.	$\text{mean} = \frac{85+86}{2}$
	$\text{mean} = 85.5$
	Kristen's median score is 85.5.

Another measure of the center is the mode. The **mode** is the most frequent value. The **frequency**, is the number of times a number occurs. So the mode of a set of numbers is the number with the highest frequency. There can be

more than one mode in a data set as long as those values have the same frequency and that frequency is the highest. A data set with two modes is called bimodal.

Note:
The **mode** of a set of numbers is the number with the highest frequency.

Note:

List the data values in numerical order.
Count the number of times each value appears.
The mode is the value with the highest frequency.

Example:
Finding the Mode
Statistics exam scores for 20 students are as follows:
50535959636372727272727678818384848493
Exercise:

Problem: Find the mode.

Solution:

The most frequent score is 72, which occurs five times. Mode = 72.

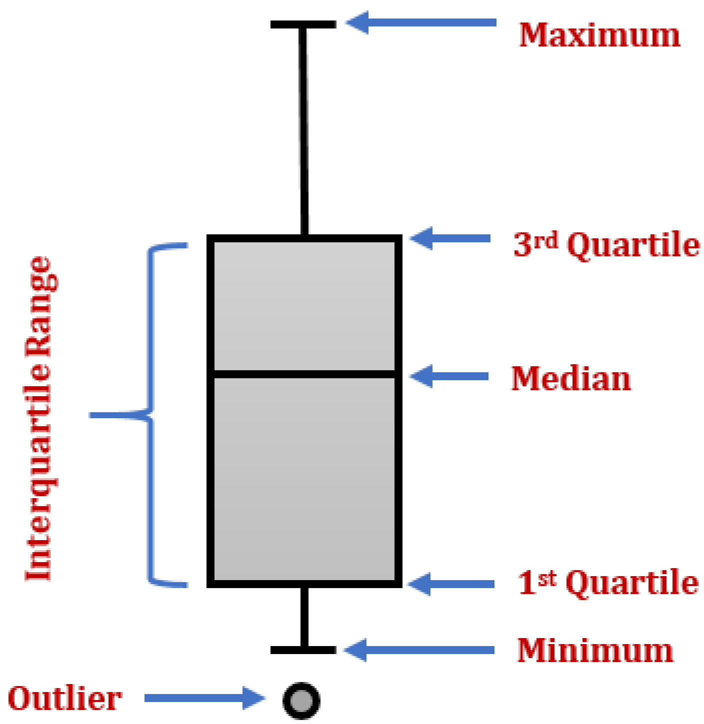
The Five-Number Summary [footnote]

Section material derived by Amanda Towry

The **five-number summary** is a set of measurements about the central tendencies of a set of data. These summaries are used to create boxplots which allow a good analysis of the spread of the data.

Note: A description of the spread of a data set which consists of the following aspects:

- Minimum Value
- First Quartile
- Median
- Third Quartile
- Maximum Value



Credit: Created by Amanda Towry by use of Microsoft Word

Box Plots [footnote]

Section material derived from Openstax Introductory Statistics: Descriptive Statistics-Box Plots

Box plots (also called **box-and-whisker plots** or **box-whisker plots**) give a good graphical image of the concentration of the data. They also show how far the extreme values are from most of the data. A box plot is constructed from five values: the minimum value, the first quartile, the median, the third

quartile, and the maximum value. We use these values to compare how close other data values are to them.

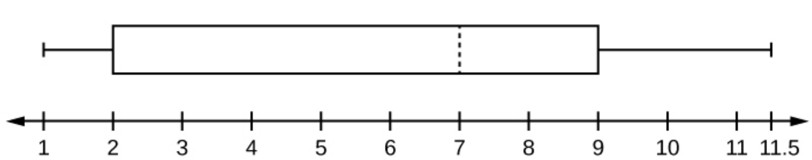
To construct a box plot, use a horizontal or vertical number line and a rectangular box. The smallest and largest data values label the endpoints of the axis. The first quartile marks one end of the box and the third quartile marks the other end of the box. Quartiles are determined by splitting the data in half around its median value. The median of the "lower half" of the data will give you your first quartile. The median of the "upper half" of the split data will give you your third quartile. Approximately **the middle 50 percent of the data fall inside the box.** The "whiskers" extend from the ends of the box to the smallest and largest data values. The median or second quartile can be between the first and third quartiles, or it can be one, or the other, or both. The box plot gives a good, quick picture of the data.

Note:
You may encounter box-and-whisker plots that have dots marking outlier values. In those cases, the whiskers are not extending to the minimum and maximum values.

Consider, again, this dataset.

1 1 2 2 4 6 6.8 7.2 8 8.3 9 10 10 11.5

The first quartile is two, the median is seven, and the third quartile is nine. The smallest value is one, and the largest value is 11.5. The following image shows the constructed box plot.



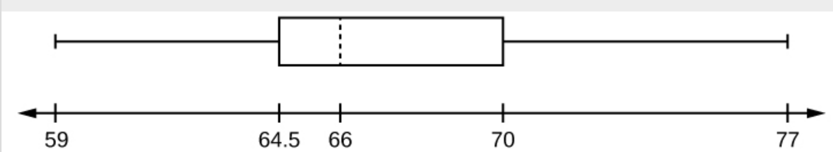
The two whiskers extend from the first quartile to the smallest value and from the third quartile to the largest value. The median is shown with a

dashed line.

Note:
It is important to start a box plot with a **scaled number line**. Otherwise the box plot may not be useful.

Example:
Creating a Boxplot
The following data are the heights of 40 students in a statistics class.
59 60 61 62 62 63 63 64 64 64 65 65 65 65 65 65 65 65 65 66 66 67 67 68 68 69 70 70 70 70 70 71 71 72 72 73 74 74 75 77
Construct a box plot with the following properties; the calculator intructions for the minimum and maximum values as well as the quartiles follow the example.

- Minimum value = 59
- Maximum value = 77
- $Q1$: First quartile = 64.5
- $Q2$: Second quartile or median= 66
- $Q3$: Third quartile = 70



a. Each quarter has approximately 25% of the data.
b. The spreads of the four quarters are 64.5 – 59 = 5.5 (first quarter), 66 – 64.5 = 1.5 (second quarter), 70 – 66 = 4 (third quarter), and 77 – 70 = 7 (fourth quarter). So, the second quarter has the smallest spread and the fourth quarter has the largest spread.
c. Range = maximum value – the minimum value = 77 – 59 = 18
d. Interquartile Range: $IQR = Q3 – Q1 = 70 – 64.5 = 5.5$.
e. The interval 59–65 has more than 25% of the data so it has more data in it than the interval 66 through 70 which has 25% of the data.

f. The middle 50% (middle half) of the data has a range of 5.5 inches.

Key Concepts

- **Calculate the mean of a set of numbers.**

  Write the formula for the mean $\text{mean} = \frac{\text{sum of values in data set}}{n}$
  Find the sum of all the values in the set. Write the sum in the numerator.
  Count the number, n, of values in the set. Write this number in the denominator.
  Simplify the fraction.
  Check to see that the mean is reasonable. It should be greater than the least number and less than the greatest number in the set.

- **Find the median of a set of numbers.**

  List the numbers from least to greatest.
  Count how many numbers are in the set. Call this n.

Is n odd or even?	If n is an odd number, the median is the middle value.	If n is an even number, the median is the mean of the two middle values

- **Identify the mode of a set of numbers.**

  List the data values in numerical order.
  Count the number of times each value appears.
  The mode is the value with the highest frequency.

Glossary

Box plot
: a graph that gives a quick picture of the middle 50% of the data

First Quartile

the value that is the median of the of the lower half of the ordered data set.

Mean
: is the arithmetic average of the numbers. The formula is $\text{mean} = \frac{\text{sum of values in data set}}{n}$

Median
: The median of a set of data values is the middle value.

- Half the data values are less than or equal to the median.
- Half the data values are greater than or equal to the median.

Mode
: The mode of a set of numbers is the number with the highest frequency.

Qualitative Data
: a set of observations, or possible outcomes, whose value is indicated by a label.

Quantitative Data
: a set of observations, or possible outcomes, whose value is indicated by a number.

Random Sampling
: a method of selecting a sample that gives every member of the population an equal chance of being selected.

Third Quartile
: the value that is the median of the of the upper half of the ordered data set

Measuring the Spread of Data: Lessons 3.A - 3.C
This module works with Lessons 3A-3C of our Corequiste MAT 1043 (QR), and NCBO.

Note:
In case you missed something in class, or just want to review a specific topic covered in this Module, here is a list of topics covered:

Data Sampling [footnote]

Section material derived from Openstax Introductory Statistics: Sampling and Data-Data Sampling and Variation in Data and Sampling

In statistics, we generally want to study a population. You can think of a population as a collection of persons, things, or objects under study. Gathering information about an entire population often costs too much or is virtually impossible. Instead, we use a **sample** of the population. **A sample should have the same characteristics as the population it is representing**. The idea of sampling is to select a portion (or subset) of the larger population and study that portion (the sample) to gain information about the population. Data are the result of sampling from a population.

Most statisticians use various methods of random sampling in an attempt to achieve this goal. A random sample is a representative group from the

population chosen by using a method that gives each individual in the population an equal chance of being included in the sample. There are several different methods of **random sampling**. The easiest method to describe is called a **simple random sample**. Any group of *n* individuals is equally likely to be chosen as any other group of *n* individuals if the simple random sampling technique is used. In other words, each sample of the same size has an equal chance of being selected.

If we were to examine two samples representing the same population, even if we used random sampling methods for the samples, they would not be exactly the same. Just as there is variation in data, there is variation in samples. As you become accustomed to sampling, the variability will begin to seem natural.

Samples that contain different individuals result in different data. This is true even when the samples are well-chosen and representative of the population. When properly selected, larger samples model the population more closely than smaller samples. There are many different potential problems that can affect the reliability of a sample. Statistical data needs to be critically analyzed, not simply accepted.

From the sample data, we can calculate a **statistic**. A statistic is a number that represents a property of the sample. For example, if we consider one math class to be a sample of the population of all math classes, then the average number of points earned by students in that one math class at the end of the term is an example of a statistic. The statistic is an estimate of a population parameter. A parameter is a numerical characteristic of the whole population that can be estimated by a statistic. Since we considered all math classes to be the population, then the average number of points earned per student over all the math classes is an example of a parameter.

The Central Limit Theorem [footnote]

Section material derived from Openstax Introductory Statistics: The Central Limit Theorem-The Central Limit Theorem for Sample Means Averages and Introductory Statistics: Descriptive Statistics-Measures of the Center of the Data

Remember, from Measures of Central Tendancies, the mean of a set of data is the result of dividing the sum of the values by the number of values.

There are two types of means that we will analyze: the **sample mean** and the **population mean**. As is implied by their names, the population mean is the mean of the population being investigated. The sample mean is the mean of the sample taken from the population being investigated.

The letter used to represent the **sample mean** is an *x* with a bar over it (pronounced "*x* bar"): x.

The Greek letter μ (pronounced "mew") represents the **population mean**. One of the requirements for the **sample mean** to be a good estimate of the **population mean** is for the sample taken to be truly random.

Note: The Law of Large Numbers says that if you take samples of larger and larger size from any population, then the mean x of the sample is very likely to get closer and closer to μ. This is discussed in more detail later in the text.

The **central limit theorem** for sample means says that if you keep drawing larger and larger samples (such as rolling one, two, five, and finally, ten dice) and **calculating their means,** the sample means form their own **normal distribution** (the sampling distribution). The normal distribution has the same mean as the original distribution and a variance that equals the original variance divided by the sample size. Standard deviation is the square root of variance, so the standard deviation of the sampling distribution is the standard deviation of the original distribution divided by the square root of *n*. The variable *n* is the number of values that are averaged together, not the number of times the experiment is done.

To put it more formally, if you draw random samples of size *n*, the distribution of the random variable X, which consists of sample means, is

called the **sampling distribution of the mean**. The sampling distribution of the mean approaches a normal distribution as *n*, the **sample size**, increases.

The Standard Deviation [footnote]

Section material derived from Openstax Introductory Statistics: Descriptive Statistics-Measures of the Spread of the Data and Statistics 1: Descriptive Statistics-Measures of the Spread of the Data

An important characteristic of any set of data is the variation in the data. In some data sets, the data values are concentrated closely near the mean; in other data sets, the data values are more widely spread out from the mean. The most common measure of variation, or spread, is the standard deviation. The **standard deviation** is a number that measures how far data values are from their mean.

The Standard Deviation

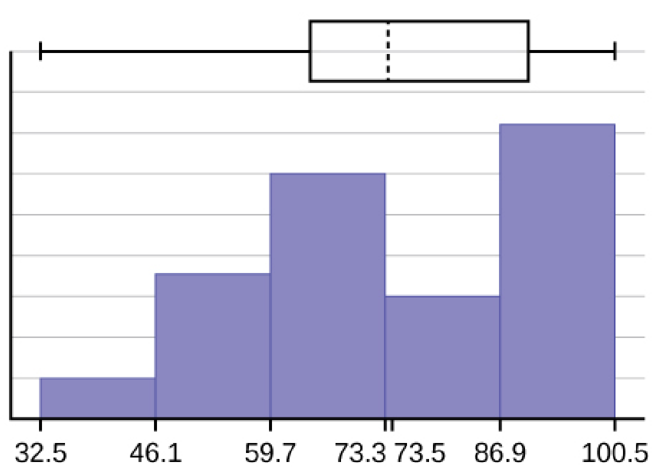
- provides a numerical measure of the overall amount of variation in a data set.
- shows how spread out the data are about the mean.
- is always positive or zero.
- is small when the data are all concentrated close to the mean, exhibiting little variation or spread and is larger when the data values are more spread out from the mean, exhibiting more variation.
- a positive deviation occurs when the data value is greater than the mean, whereas a negative deviation occurs when the data value is less than the mean.
- can be used to determine whether a particular data value is close to or far from the mean.
- **If you add the deviations, the sum is always zero**. So you cannot simply add the deviations to get the spread of the data.

Note: Your concentration should be on what the standard deviation tells us about the data. The standard deviation is a number which measures how far

the data are spread from the mean. Let a calculator or computer do the arithmetic.

The standard deviation, *s* or *σ*, is either zero or larger than zero. Describing the data with reference to the spread is called "variability". The variability in data depends upon the method by which the outcomes are obtained; for example, by measuring or by random sampling. When the standard deviation is zero, there is no spread; that is, the all the data values are equal to each other. The standard deviation is small when the data are all concentrated close to the mean, and is larger when the data values show more variation from the mean. When the standard deviation is a lot larger than zero, the data values are very spread out about the mean; outliers can make *s* or *σ* very large.

The standard deviation, when first presented, can seem unclear. By graphing your data, you can get a better "feel" for the deviations and the standard deviation. You will find that in symmetrical distributions, the standard deviation can be very helpful but in skewed distributions, the standard deviation may not be much help. The reason is that the two sides of a skewed distribution have different spreads. In a skewed distribution, it is better to look at the first quartile, the median, the third quartile, the smallest value, and the largest value. Because numbers can be confusing, **always graph your data**. Display your data in a histogram or a box plot.



The following lists give a few facts that provide a little more insight into what the standard deviation tells us about the distribution of the data.

- At least 75% of the data is within two standard deviations of the mean.
- At least 89% of the data is within three standard deviations of the mean.
- At least 95% of the data is within 4.5 standard deviations of the mean.

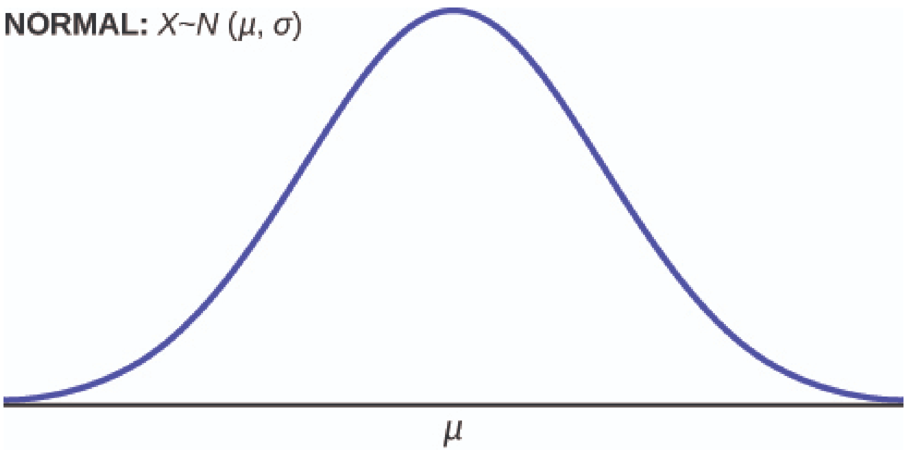
- Approximately 68% of the data is within one standard deviation of the mean.
- Approximately 95% of the data is within two standard deviations of the mean.
- More than 99% of the data is within three standard deviations of the mean.
- This is known as the Empirical Rule.
- It is important to note that this rule only applies when the shape of the distribution of the data is bell-shaped and symmetric. We will learn more about this when studying the "Normal" or "Gaussian" probability distribution in later chapters.

The Normal Distribution [footnote]

Section material derived from Openstax Introductory Statistics: The Normal Distribution-Introduction and Introductory Statistics: The Normal Distribution-The Standard Normal Distribution

The normal, a continuous distribution, is the most important of all the distributions. It is widely used and even more widely abused. Its graph is bell-shaped. You see the bell curve in almost all disciplines. Some of these include psychology, business, economics, the sciences, nursing, and, of course, mathematics. Some of your instructors may use the normal distribution to help determine your grade. Most IQ scores are normally distributed. Often real-estate prices fit a normal distribution. The normal distribution is extremely important, but it cannot be applied to everything in the real world.

The normal distribution has two parameters (two numerical descriptive measures): the mean (μ) and the standard deviation (σ). We can say that X is a quantity to be measured that has a normal distribution with mean (μ) and standard deviation (σ).

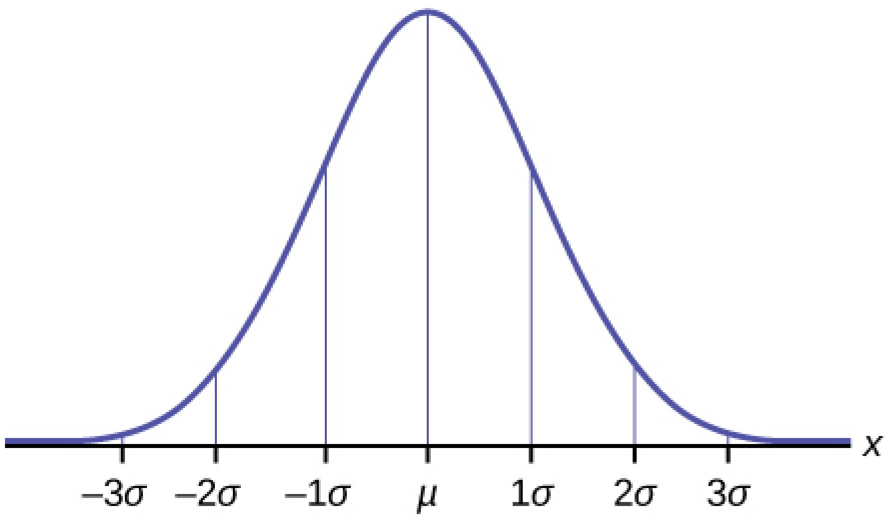


The curve is symmetric about a vertical line drawn through the mean, μ. In theory, the mean is the same as the median, because the graph is symmetric about μ. As the notation indicates, the normal distribution depends only on the mean and the standard deviation. Since the area under the curve must equal one, a change in the standard deviation, σ, causes a change in the shape of the curve; the curve becomes fatter or skinnier depending on σ. A change in μ causes the graph to shift to the left or right. This means there are an infinite number of normal probability distributions.

For a normally distributed set of data that follows this normal curve, some important facts about the spread of the data can be stated:

- About 68% of the x values lie between -1σ and $+1\sigma$ of the mean μ (within one standard deviation of the mean).
- About 95% of the x values lie between -2σ and $+2\sigma$ of the mean μ (within two standard deviations of the mean).
- About 99.7% of the x values lie between -3σ and $+3\sigma$ of the mean μ (within three standard deviations of the mean). Notice that almost all the x values lie within three standard deviations of the mean.

This allows for the normal curve to be split into quadrants or groups of data as is shown below. Where μ is the population mean, and σ is the standard deviation of the data.



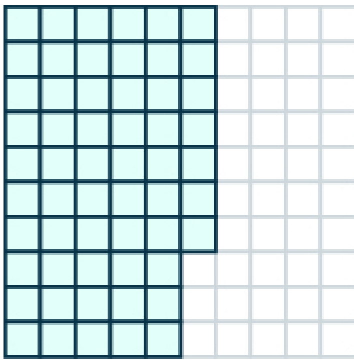
Use the Definition of Percent [footnote]

Section material derived from Openstax Prealgebra: Percents-Understand Percent

How many cents are in one dollar? There are 100 cents in a dollar. How many years are in a century? There are 100 years in a century. Does this give you a clue about what the word “percent” means? It is really two words, “per cent,” and means per one hundred. A **percent** is a ratio whose denominator is 100. We use the percent symbol $\%$, to show percent.

Note:
A percent is a ratio whose denominator is 100.

According to data from the American Association of Community Colleges (2015), about 57% of community college students are female. This means 57 out of every 100 community college students are female, as [link] shows. Out of the 100 squares on the grid, 57 are shaded, which we write as the ratio $\frac{57}{100}$.



Among every 100 community college students, 57 are female.

Similarly, 25% means a ratio of $\frac{25}{100}$, 3% means a ratio of $\frac{3}{100}$ and 100% means a ratio of $\frac{100}{100}$. In words, "one hundred percent" means the total 100% is $\frac{100}{100}$, and since $\frac{100}{100} = 1$, we see that 100% means 1 whole.

Example:
Converting from Percent to Fraction
Exercise:

Problem:

According to the Public Policy Institute of California (2010), 44% of parents of public school children would like their youngest child to earn a graduate degree. Write this percent as a ratio.

Solution:
Solution

The amount we want to convert is 44%.	44%
Write the percent as a ratio. Remember that *percent* means per 100.	$\frac{44}{100}$

Example:
Convert from Fraction to Percent
Exercise:

Problem:

In 2007, according to a U.S. Department of Education report, 21 out of every 100 first-time freshmen college students at 4-year public institutions took at least one remedial course. Write this as a ratio and then as a percent.

Solution:
Solution

The amount we want to convert is 21 out of 100.	21 out of 100
Write as a ratio.	$\frac{21}{100}$
Convert the 21 per 100 to percent.	21%

Convert Percents to Fractions and Decimals [footnote]

Section material derived from Openstax Prealgebra: Percent-Understand Percent

Since percents are ratios, they can easily be expressed as fractions. Remember that percent means per 100, so the denominator of the fraction is 100.

Note:

Write the percent as a ratio with the denominator 100.
Simplify the fraction if possible.

Example:
Convert a Percent to a Fraction
Exercise:

Problem: Convert each percent to a fraction:

ⓐ 36%

ⓑ 125%

Solution:
Solution

ⓐ	
	36%
Write as a ratio with denominator 100.	$\frac{36}{100}$
Simplify.	$\frac{9}{25}$

ⓑ	
	125%
Write as a ratio with denominator 100.	$\frac{125}{100}$
Simplify.	$\frac{5}{4}$

The previous example shows that a percent can be greater than 1. We saw that 125% means $\frac{125}{100}$, or $\frac{5}{4}$. These are improper fractions, and their values are greater than one.

Example:
Convert a Percent to a Fraction
Exercise:

Problem: Convert each percent to a fraction:

ⓐ 24.5%
ⓑ $33\frac{1}{3}\%$

Solution:
Solution

ⓐ	
	24.5%
Write as a ratio with denominator 100.	$\frac{24.5}{100}$
Clear the decimal by multiplying numerator and denominator by 10.	$\frac{24.5(10)}{100(10)}$
Multiply.	$\frac{245}{1000}$
Rewrite showing common factors.	$\frac{5\cdot 49}{5\cdot 200}$

Simplify.	$\frac{49}{200}$

ⓑ	
	$33\frac{1}{3}\%$
Write as a ratio with denominator 100.	$\frac{33\frac{1}{3}}{100}$
Write the numerator as an improper fraction.	$\frac{\frac{100}{3}}{100}$
Rewrite as fraction division, replacing 100 with $\frac{100}{1}$.	$\frac{100}{3} \div \frac{100}{1}$
Multiply by the reciprocal.	$\frac{100}{3} \cdot \frac{1}{100}$
Simplify.	$\frac{1}{3}$

In Decimals, we learned how to convert fractions to decimals. To convert a percent to a decimal, we first convert it to a fraction and then change the fraction to a decimal.

Note:

Write the percent as a ratio with the denominator 100.

Convert the fraction to a decimal by dividing the numerator by the denominator.

To convert a percent number to a decimal number, we move the decimal point two places to the left and remove the $\%$ sign. (Sometimes the decimal point does not appear in the percent number, but just like we can think of the integer 6 as 6.0, we can think of 6% as 6.0%.) Notice that we may need to add zeros in front of the number when moving the decimal to the left.

[link] uses the percents in [link] and shows visually how to convert them to decimals by moving the decimal point two places to the left.

Percent	Decimal
006.%	0.06
078.%	0.78
135.%	1.35
012.5%	0.125

Example:
Convert a Percent to a Decimal
Exercise:

Problem:

There are four suits of cards in a deck of cards—hearts, diamonds, clubs, and spades. The probability of randomly choosing a heart from a shuffled deck of cards is 25%. Convert the percent to:

ⓐ a fraction
ⓑ a decimal

(credit: Riles32807, Wikimedia Commons)

Solution:
Solution

ⓐ	
	25%
Write as a ratio with denominator 100.	$\frac{25}{100}$
Simplify.	$\frac{1}{4}$

ⓑ	$\frac{1}{4}$
Change the fraction to a decimal by dividing the numerator by the denominator.	0.25

Convert Decimals and Fractions to Percents [footnote]

Section material derived from Openstax Prealgebra: Percents-Understand Percent

To convert a decimal to a percent, remember that percent means per hundred. If we change the decimal to a fraction whose denominator is 100, it is easy to change that fraction to a percent.

Note:

Write the decimal as a fraction.
If the denominator of the fraction is not $100,$ rewrite it as an equivalent fraction with denominator $100.$
Write this ratio as a percent.

Example:
Convert a Decimal to a Percent
Exercise:

Problem: Convert each decimal to a percent: ⓐ 0.05 ⓑ 0.83

Solution:

Solution

ⓐ	
	0.05
Write as a fraction. The denominator is 100.	$\frac{5}{100}$
Write this ratio as a percent.	5%

ⓑ	
	0.83
The denominator is 100.	$\frac{83}{100}$
Write this ratio as a percent.	83%

[link] uses the decimal numbers in [link] and shows visually to convert them to percents by moving the decimal point two places to the right and then writing the $\%$ sign.

Percent	Decimal
006.%	0.06
078.%	0.78
135.%	1.35
012.5%	0.125

In Decimals, we learned how to convert fractions to decimals. Now we also know how to change decimals to percents. So to convert a fraction to a percent, we first change it to a decimal and then convert that decimal to a percent.

Note:

Convert the fraction to a decimal.
Convert the decimal to a percent.

Example:
Convert a Fraction to a Percent
Exercise:

Problem:

Convert each fraction or mixed number to a percent: ⓐ $\frac{3}{4}$ ⓑ $\frac{11}{8}$ ⓒ $2\frac{1}{5}$

Solution:
Solution

To convert a fraction to a decimal, divide the numerator by the denominator.

ⓐ	
Change to a decimal.	$\frac{3}{4}$
Write as a percent by moving the decimal two places.	0.75
	75%

ⓑ	
Change to a decimal.	$\frac{11}{8}$
Write as a percent by moving the decimal two places.	1.375
	137.5%

ⓒ	
Write as an improper fraction.	$2\frac{1}{5}$
Change to a decimal.	$\frac{11}{5}$

Write as a percent.	2.20
	220%

Notice that we needed to add zeros at the end of the number when moving the decimal two places to the right.

Sometimes when changing a fraction to a decimal, the division continues for many decimal places and we will round off the quotient. The number of decimal places we round to will depend on the situation. If the decimal involves money, we round to the hundredths place. For most other cases in this book we will round the number to the nearest thousandth, so the percent will be rounded to the nearest tenth.

Example:
Convert a Fraction to a Percent
Exercise:

Problem: Convert $\frac{5}{7}$ to a percent.

Solution:
Solution

To change a fraction to a decimal, we divide the numerator by the denominator.

	$\frac{5}{7}$
Change to a decimal—rounding to the nearest thousandth.	0.714
Write as a percent.	71.4%

When we first looked at fractions and decimals, we saw that fractions converted to a repeating decimal. When we converted the fraction $\frac{4}{3}$ to a decimal, we wrote the answer as 1.3. We will use this same notation, as well as fraction notation, when we convert fractions to percents in the next example.

Example:
Convert a Fraction to a Percent
Exercise:

Problem:

An article in a medical journal claimed that approximately $\frac{1}{3}$ of American adults are obese. Convert the fraction $\frac{1}{3}$ to a percent.

Solution:
Solution

	$\frac{1}{3}$

Change to a decimal.	$\begin{array}{r} 0.33\ldots \\ 3\overline{)1.00} \\ \underline{9} \\ 10 \\ \underline{9} \\ 1 \end{array}$
Write as a repeating decimal.	$0.333\ldots$
Write as a percent.	$33\frac{1}{3}\%$

We could also write the percent as $33.\overline{3}\%$.

Key Concepts

- **Convert a percent to a fraction.**

  Write the percent as a ratio with the denominator 100.
  Simplify the fraction if possible.

- **Convert a percent to a decimal.**

  Write the percent as a ratio with the denominator 100.
  Convert the fraction to a decimal by dividing the numerator by the denominator.

- **Convert a decimal to a percent.**

  Write the decimal as a fraction.
  If the denominator of the fraction is not 100, rewrite it as an equivalent fraction with denominator 100.
  Write this ratio as a percent.

- **Convert a fraction to a percent.**

Convert the fraction to a decimal.
Convert the decimal to a percent.

- **Add or subtract decimals.**

  Write the numbers vertically so the decimal points line up.
  Use zeros as place holders, as needed.
  Add or subtract the numbers as if they were whole numbers. Then place the decimal in the answer under the decimal points in the given numbers.

- **Multiply decimal numbers.**

  Determine the sign of the product.
  Write the numbers in vertical format, lining up the numbers on the right.
  Multiply the numbers as if they were whole numbers, temporarily ignoring the decimal points.
  Place the decimal point. The number of decimal places in the product is the sum of the number of decimal places in the factors. If needed, use zeros as placeholders.
  Write the product with the appropriate sign.

- **Multiply a decimal by a power of 10.**

  Move the decimal point to the right the same number of places as the number of zeros in the power of 10.
  Write zeros at the end of the number as placeholders if needed.

- **Divide a decimal by a whole number.**

  Write as long division, placing the decimal point in the quotient above the decimal point in the dividend.
  Divide as usual.

- **Divide decimal numbers.**

  Determine the sign of the quotient.

Make the divisor a whole number by moving the decimal point all the way to the right. Move the decimal point in the dividend the same number of places to the right, writing zeros as needed.
Divide. Place the decimal point in the quotient above the decimal point in the dividend.
Write the quotient with the appropriate sign.

Glossary

Continuous Data
: a variable whose outcomes are measured

Discrete Data
: a variable whose outcomes are counted

Percent
: a ratio whose denominator is 100

Population
: all individuals, objects, or measurements whose properties are being studied.

Qualitative Data
: a set of observations, or possible outcomes, whose value is indicated by a label.

Quantitative Data
: a set of observations, or possible outcomes, whose value is indicated by a number.

Sample
: a subset of the population studied.

Standard Deviation
: a number that is equal to the square root of the variance and measures how far data values are from their mean; notation: s for sample standard deviation and σ for population standard deviation.

Statistic

a numerical characteristic of the sample; a statistic estimates the corresponding population parameter.

Variable

a characteristic of interest for each person or object in a population.

Fractions: Lessons 4.A - 4.B
By the end of this section, you will be able to:

- Simplify Fractions
- Find LCD
- Add and Subtract Fractions
- Multiply and Divide Fractions
- Find reciprovals
- Convert Fractions

Note:
In case you missed something in class, or just want to review a specific topic covered in this Module, here is a list of topics covered:

1. Simplifying Fractions [link]
2. Least Common Denominator [link]
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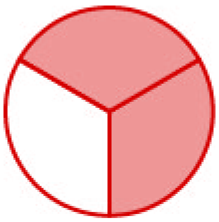
Simplifying Fractions [footnote]

Section material derived from Openstax Prealgebra: Fractions-Multiply and Divide Fractions

A **fraction** is a way to represent parts of a whole. The fraction $\frac{2}{3}$ represents two of three equal parts. See [link]. In the fraction $\frac{2}{3}$, the 2 is called the

numerator and the 3 is called the **denominator**. The line is called the fraction bar.



In the circle, $\frac{2}{3}$ of the circle is shaded—2 of the 3 equal parts.

Note:
A **fraction** is written $\frac{a}{b}$, where $b \neq 0$ and
a is the **numerator** and b is the **denominator**.
A fraction represents parts of a whole. The denominator b is the number of equal parts the whole has been divided into, and the numerator a indicates how many parts are included.

Fractions that have the same value are **equivalent fractions**. The
Equivalent Fractions

Property allows us to find equivalent fractions and also simplify fractions.

Note:
If a, b, and c are numbers where $b \neq 0, c \neq 0$,
then $\frac{a}{b} = \frac{a \cdot c}{b \cdot c}$ and $\frac{a \cdot c}{b \cdot c} = \frac{a}{b}$.

A fraction is considered simplified if there are no common factors, other than 1, in its numerator and denominator.

For example,

$\frac{2}{3}$ is simplified because there are no common factors of 2 and 3.

$\frac{10}{15}$ is not simplified because 5 is a common factor of 10 and 15.

We simplify, or reduce, a fraction by removing the common factors of the numerator and denominator. A fraction is not simplified until all common factors have been removed. If an expression has fractions, it is not completely simplified until the fractions are simplified.

Sometimes it may not be easy to find common factors of the numerator and denominator. When this happens, a good idea is to factor the numerator and the denominator into prime numbers. Then divide out the common factors using the Equivalent Fractions Property.

Example:
How To Simplify a Fraction
Exercise:

Problem: Simplify: $-\frac{315}{770}$.

Solution:

Step 1. Rewrite the numerator and denominator to show the common factors. If needed, use a factor tree.	Rewrite 315 and 770 as the product of the primes.	$-\frac{315}{770}$ $-\frac{3\cdot 3\cdot 5\cdot 7}{2\cdot 5\cdot 7\cdot 11}$
Step 2. Simplify using the equivalent fractions property by dividing out common factors.	Mark the common factors 5 and 7. Divide out the common factors	$-\frac{3\cdot 3\cdot \cancel{5}\cdot \cancel{7}}{2\cdot \cancel{5}\cdot \cancel{7}\cdot 11}$ $-\frac{3\cdot 3}{2\cdot 11}$
Step 3. Multiply the remaining factors, if necessary.		$-\frac{9}{22}$

We now summarize the steps you should follow to simplify fractions.

Note:

Rewrite the numerator and denominator to show the common factors. If needed, factor the numerator and denominator into prime numbers first.
Simplify using the Equivalent Fractions Property by dividing out common factors.
Multiply any remaining factors.

Least Common Denominator [footnote]

Section material derived from Openstax Prealgebra: Fractions-Add and Subtract Fractions with Different Denominators

In the previous section, we explained how to add and subtract fractions with a common denominator. But how can we add and subtract fractions with unlike denominators?

Let’s think about coins again. Can you add one quarter and one dime? You could say there are two coins, but that’s not very useful. To find the total value of one quarter plus one dime, you change them to the same kind of unit—cents. One quarter equals 25 cents and one dime equals 10 cents, so the sum is 35 cents. See [link].

Together, a quarter and a dime are worth 35 cents, or $\frac{35}{100}$ of a dollar.

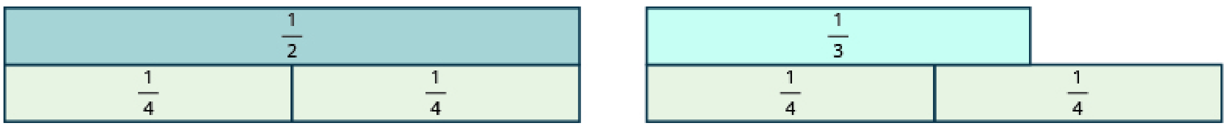
Similarly, when we add fractions with different denominators we have to convert them to equivalent fractions with a common denominator. With the coins, when we convert to cents, the denominator is 100. Since there are 100 cents in one dollar, 25 cents is $\frac{25}{100}$ and 10 cents is $\frac{10}{100}$. So we add $\frac{25}{100}+\frac{10}{100}$ to get $\frac{35}{100}$, which is 35 cents.

You have practiced adding and subtracting fractions with common denominators. Now let’s see what you need to do with fractions that have different denominators.

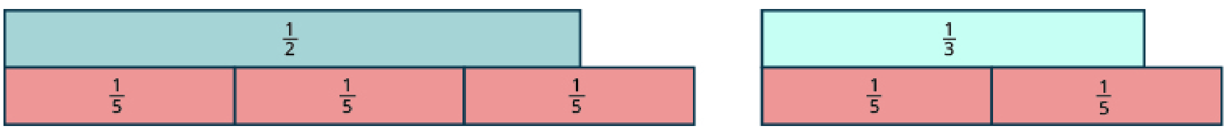
First, we will use fraction tiles to model finding the common denominator of $\frac{1}{2}$ and $\frac{1}{3}$.

We'll start with one $\frac{1}{2}$ tile and $\frac{1}{3}$ tile. We want to find a common fraction tile that we can use to match *both* $\frac{1}{2}$ and $\frac{1}{3}$ exactly.

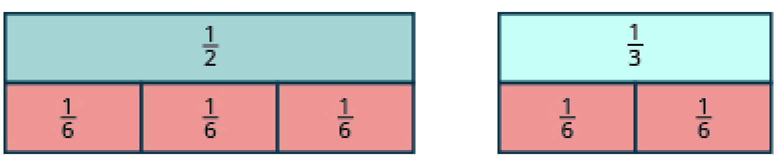
If we try the $\frac{1}{4}$ pieces, 2 of them exactly match the $\frac{1}{2}$ piece, but they do not exactly match the $\frac{1}{3}$ piece.



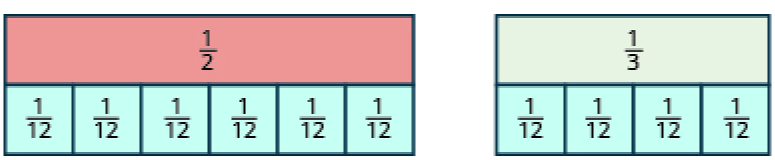
If we try the $\frac{1}{5}$ pieces, they do not exactly cover the $\frac{1}{2}$ piece or the $\frac{1}{3}$ piece.



If we try the $\frac{1}{6}$ pieces, we see that exactly 3 of them cover the $\frac{1}{2}$ piece, and exactly 2 of them cover the $\frac{1}{3}$ piece.



If we were to try the $\frac{1}{12}$ pieces, they would also work.



Even smaller tiles, such as $\frac{1}{24}$ and $\frac{1}{48}$, would also exactly cover the $\frac{1}{2}$ piece and the $\frac{1}{3}$ piece.

The denominator of the largest piece that covers both fractions is the **least common denominator (LCD)** of the two fractions. So, the least common denominator of $\frac{1}{2}$ and $\frac{1}{3}$ is 6.

Notice that all of the tiles that cover $\frac{1}{2}$ and $\frac{1}{3}$ have something in common: Their denominators are common multiples of 2 and 3, the denominators of $\frac{1}{2}$ and $\frac{1}{3}$. The least common multiple (LCM) of the denominators is 6, and so we say that 6 is the least common denominator (LCD) of the fractions $\frac{1}{2}$ and $\frac{1}{3}$.

Note: Doing the Manipulative Mathematics activity "Finding the Least Common Denominator" will help you develop a better understanding of the LCD.

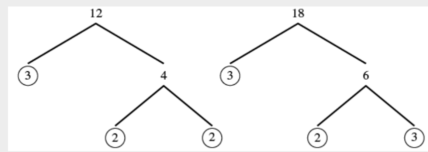
Note:
Least Common Denominator
The **least common denominator (LCD)** of two fractions is the least common multiple (LCM) of their denominators.

To find the LCD of two fractions, we will find the LCM of their denominators. We follow the procedure we used earlier to find the LCM of two numbers. We only use the denominators of the fractions, not the numerators, when finding the LCD.

Example:
Finding the LCD of Two Fractions
Exercise:

Problem: Find the LCD for the fractions $\frac{7}{12}$ and $\frac{5}{18}$.

Solution:
Solution

Factor each denominator into its primes.	
List the primes of 12 and the primes of 18 lining them up in columns when possible.	$12 = 2 \cdot 2 \cdot 3$ $18 = 2 \cdot \quad 3 \cdot 3$
Bring down the columns.	$12 = 2 \cdot 2 \cdot 3$ $18 = 2 \cdot \quad 3 \cdot 3$ $\text{LCM} = 2 \cdot 2 \cdot 3 \cdot 3$
Multiply the factors. The product is the LCM.	$\text{LCM} = 36$
The LCM of 12 and 18 is 36, so the LCD of $\frac{7}{12}$ and $\frac{5}{18}$ is 36.	LCD of $\frac{7}{12}$ and $\frac{5}{18}$ is 36.

To find the LCD of two fractions, find the LCM of their denominators. Notice how the steps shown below are similar to the steps we took to find the LCM.

Note:

Factor each denominator into its primes.
List the primes, matching primes in columns when possible.
Bring down the columns.
Multiply the factors. The product is the LCM of the denominators.
The LCM of the denominators is the LCD of the fractions.

Example:
Finding the LCD of Two Fractions
Exercise:

Problem:

Find the least common denominator for the fractions $\frac{8}{15}$ and $\frac{11}{24}$.

Solution:
Solution

To find the LCD, we find the LCM of the denominators.

Find the LCM of 15 and 24.

$$\begin{array}{rl} 15 = & \phantom{2\cdot2\cdot2\cdot{}} 3\cdot5 \\ 24 = & 2\cdot2\cdot2\cdot3 \\ \hline \text{LCD} = & 2\cdot2\cdot2\cdot3\cdot5 \\ \text{LCD} = & 120 \end{array}$$

The LCM of 15 and 24 is 120. So, the LCD of $\frac{8}{15}$ and $\frac{11}{24}$ is 120.

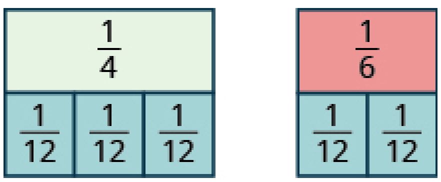
Convert to Equivalent Fractions [footnote]

Section material derived from Openstax Prealgebra: Fractions-Add and Subtract Fractions with Different Denominators

Earlier, we used fraction tiles to see that the LCD of $\frac{1}{4}$ and $\frac{1}{6}$ is 12. We saw that three $\frac{1}{12}$ pieces exactly covered $\frac{1}{4}$ and two $\frac{1}{12}$ pieces exactly covered $\frac{1}{6}$, so

Equation:

$$\frac{1}{4} = \frac{3}{12} \text{and} \frac{1}{6} = \frac{2}{12}.$$



We say that $\frac{1}{4}$ and $\frac{3}{12}$ are equivalent fractions and also that $\frac{1}{6}$ and $\frac{2}{12}$ are equivalent fractions.

We can use the Equivalent Fractions Property to algebraically change a fraction to an equivalent one. Remember, two fractions are equivalent if they have the same value. The Equivalent Fractions Property is repeated below for reference.

Note:
If a, b, c are whole numbers where $b \neq 0, c \neq 0,$ then
Equation:

$$\frac{a}{b} = \frac{a \cdot c}{b \cdot c} \quad \text{and} \quad \frac{a \cdot c}{b \cdot c} = \frac{a}{b}$$

To add or subtract fractions with different denominators, we will first have to convert each fraction to an equivalent fraction with the LCD. Let’s see how to change $\frac{1}{4}$ and $\frac{1}{6}$ to equivalent fractions with denominator 12 without using models.

Example:
Coverting to Equivalent Fractions
Exercise:

Problem:

Convert $\frac{1}{4}$ and $\frac{1}{6}$ to equivalent fractions with denominator 12, their LCD.

Solution:
Solution

Find the LCD.	The LCD of $\frac{1}{4}$ and $\frac{1}{6}$ is 12.
Find the number to multiply 4 to get 12.	$4 \cdot 3 = 12$
Find the number to multiply 6 to get 12.	$6 \cdot 2 = 12$
Use the Equivalent Fractions Property to convert	

each fraction to an equivalent fraction with the LCD, multiplying both the numerator and denominator of each fraction by the same number.	$\frac{1}{4}$ $\frac{1}{6}$ $\frac{1\cdot 3}{4\cdot 3}$ $\frac{1\cdot 2}{6\cdot 2}$
Simplify the numerators and denominators.	$\frac{3}{12}$ $\frac{2}{12}$

We do not reduce the resulting fractions. If we did, we would get back to our original fractions and lose the common denominator.

Note:
Convert two fractions to equivalent fractions with their LCD as the common denominator.

Find the LCD.
For each fraction, determine the number needed to multiply the denominator to get the LCD.
Use the Equivalent Fractions Property to multiply both the numerator and denominator by the number you found in Step 2.
Simplify the numerator and denominator.

Example:
Coverting to Equivalent Fractions with an LCD
Exercise:

Problem:

Convert $\frac{8}{15}$ and $\frac{11}{24}$ to equivalent fractions with denominator 120, their LCD.

Solution:
Solution

	The LCD is 120. We will start at Step 2.
Find the number that must multiply 15 to get 120.	$15 \cdot 8 = 120$
Find the number that must multiply 24 to get 120.	$24 \cdot 5 = 120$
Use the Equivalent Fractions Property.	$\frac{8 \cdot 8}{15 \cdot 8}$ $\frac{11 \cdot 5}{24 \cdot 5}$
Simplify the numerators and denominators.	$\frac{64}{120}$ $\frac{55}{120}$

Add and Subtract Fractions [footnote]

Section material derived from Openstax Prealgebra: Fractions-Add and Subtract Fractions with Common Denominators

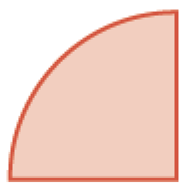
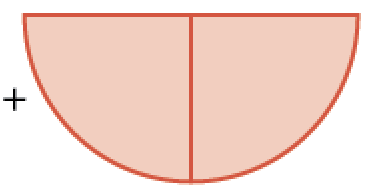
How many quarters are pictured? One quarter plus 2 quarters equals 3 quarters.

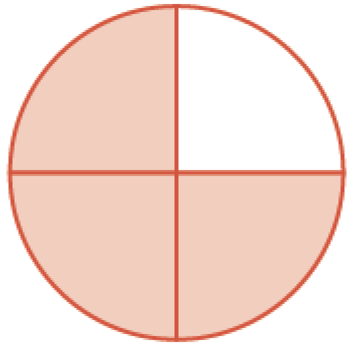
Remember, quarters are really fractions of a dollar. Quarters are another way to say fourths. So the picture of the coins shows that

Equation:

$$\overset{\frac{1}{4}}{\text{one quarter}} + \overset{\frac{2}{4}}{\text{two quarters}} = \overset{\frac{3}{4}}{\text{three quarters}}$$

Let’s use fraction circles to model the same example, $\frac{1}{4} + \frac{2}{4}$.

Start with one $\frac{1}{4}$ piece.		$\frac{1}{4}$
Add two more $\frac{1}{4}$ pieces.	+ 	$+\ \frac{2}{4}$
The result is $\frac{3}{4}$.		

		$\frac{3}{4}$

So again, we see that

Equation:

$$\frac{1}{4} + \frac{2}{4} = \frac{3}{4}$$

Example:

Adding Fractions

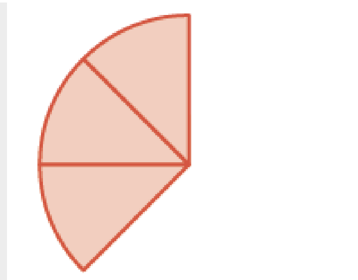
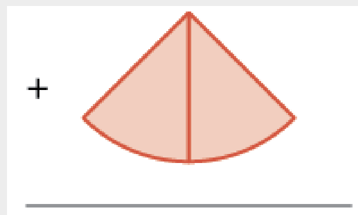
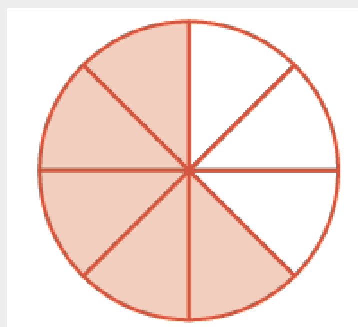
Exercise:

Problem: Use a model to find the sum $\frac{3}{8} + \frac{2}{8}$.

Solution:

Solution

Start with three $\frac{1}{8}$ pieces.		

		$\frac{3}{8}$
Add two $\frac{1}{8}$ pieces.		$+\frac{2}{8}$
How many $\frac{1}{8}$ pieces are there?		$\frac{5}{8}$

There are five $\frac{1}{8}$ pieces, or five-eighths. The model shows that $\frac{3}{8}+\frac{2}{8}=\frac{5}{8}$.

Note:
If a, b, and c are numbers where $c \neq 0$, then
Equation:

$$\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c} \text{ and } \frac{a}{c}-\frac{b}{c}=\frac{a-b}{c}$$

To add or subtract fractions, add or subtract the numerators and place the result over the common denominator.

Note:

Do they have a common denominator?

- Yes—go to step 2.
- No—rewrite each fraction with the LCD (least common denominator).
  - Find the LCD.
  - Change each fraction into an equivalent fraction with the LCD as its denominator.

Add or subtract the fractions.
Simplify, if possible.

Adding and Subtracting Fractions by Finding Factors and LCDs [footnote]

Section material derived from Openstax Prealgebra: Fractions-Add and Subtract Fractions with Different Denominators

Once we have converted two fractions to equivalent forms with common denominators, we can add or subtract them by adding or subtracting the numerators.

Note:

Find the LCD.

Convert each fraction to an equivalent form with the LCD as the denominator.
Add or subtract the fractions.
Write the result in simplified form.

Example:
Adding Fractions
Exercise:

Problem: Add: $\frac{1}{2} + \frac{1}{3}$.

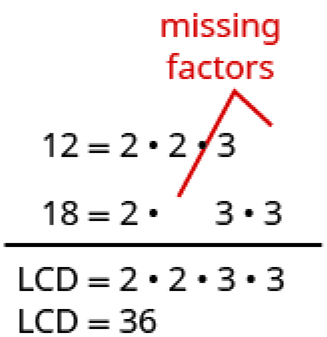
Solution:
Solution

	$\frac{1}{2} + \frac{1}{3}$
Find the LCD of 2, 3. $$\begin{aligned} 2 &= 2 \\ 3 &= \quad 3 \\ \hline \text{LCD} &= 2 \cdot 3 \\ \text{LCD} &= 6 \end{aligned}$$	
Change into equivalent fractions with the LCD 6.	$\frac{1 \cdot 3}{2 \cdot 3} + \frac{1 \cdot 2}{3 \cdot 2}$

Simplify the numerators and denominators.	$\frac{3}{6}+\frac{2}{6}$
Add.	$\frac{5}{6}$

Remember, always check to see if the answer can be simplified. Since 5 and 6 have no common factors, the fraction $\frac{5}{6}$ cannot be reduced.

When we use the Equivalent Fractions Property, there is a quick way to find the number you need to multiply by to get the LCD. Write the factors of the denominators and the LCD just as you did to find the LCD. The "missing" factors of each denominator are the numbers you need.



The LCD, 36, has 2 factors of 2 and 2 factors of 3.

Twelve has two factors of 2, but only one of 3—so it is 'missing' one 3. We multiplied the numerator and denominator of $\frac{7}{12}$ by 3 to get an equivalent fraction with denominator 36.

Eighteen is missing one factor of 2—so you multiply the numerator and denominator $\frac{5}{18}$ by 2 to get an equivalent fraction with denominator 36. We will apply this method as we subtract the fractions in the next example.

Example:

Subtracting Fractions
Exercise:

Problem: Subtract: $\frac{7}{15} - \frac{19}{24}$.

Solution:
Solution

	$\frac{7}{15} - \frac{19}{24}$
Find the LCD. $\begin{array}{l} \quad 15 = \qquad\qquad 3 \cdot 5 \\ \quad 24 = 2 \cdot 2 \cdot 2 \cdot 3 \\ \hline \text{LCD} = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5 \\ \text{LCD} = 120 \end{array}$ 15 is 'missing' three factors of 2 24 is 'missing' a factor of 5	
Rewrite as equivalent fractions with the LCD.	$\frac{7 \cdot 8}{15 \cdot 8} - \frac{19 \cdot 5}{24 \cdot 5}$
Simplify each numerator and denominator.	$\frac{56}{120} - \frac{95}{120}$
Subtract.	$-\frac{39}{120}$
Rewrite showing the common factor of 3.	$-\frac{13 \cdot 3}{40 \cdot 3}$

Remove the common factor to simplify.	$-\frac{13}{40}$

Example:
Adding with Negative Fractions
Exercise:

Problem: Add: $-\frac{11}{30}+\frac{23}{42}$.

Solution:
Solution

	$-\frac{11}{30}+\frac{23}{42}$
Find the LCD. $$\begin{aligned} 30 &= 2\cdot3\cdot5 \\ 42 &= 2\cdot3\cdot7 \\ \hline \text{LCD} &= 2\cdot3\cdot5\cdot7 \\ \text{LCD} &= 210 \end{aligned}$$	
Rewrite as equivalent fractions with the LCD.	$-\frac{11\cdot7}{30\cdot7}+\frac{23\cdot5}{42\cdot5}$
Simplify each numerator and denominator.	$-\frac{77}{210}+\frac{115}{210}$
Add.	

	$\frac{38}{210}$
Rewrite showing the common factor of 2.	$\frac{19 \cdot 2}{105 \cdot 2}$
Remove the common factor to simplify.	$\frac{19}{105}$

In the next example, one of the fractions has a variable in its numerator. We follow the same steps as when both numerators are numbers.

Note:
ACCESS ADDITIONAL ONLINE RESOURCES

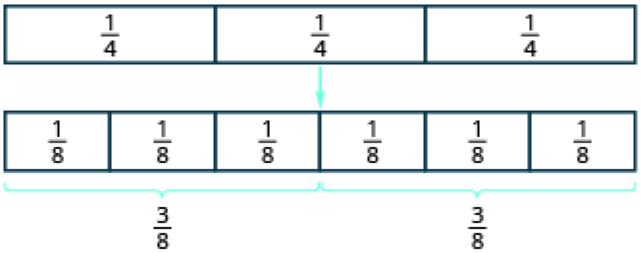
- Adding Mixed Numbers
- Subtracting Mixed Numbers

Multiply fractions [footnote]

Section material derived from Openstax Prealgebra: Fractions-Multiply and Divide Fractions

A model may help you understand multiplication of fractions. We will use fraction tiles to model $\frac{1}{2} \cdot \frac{3}{4}$. To multiply $\frac{1}{2}$ and $\frac{3}{4}$, think $\frac{1}{2}$ of $\frac{3}{4}$.

Start with fraction tiles for three-fourths. To find one-half of three-fourths, we need to divide them into two equal groups. Since we cannot divide the three $\frac{1}{4}$ tiles evenly into two parts, we exchange them for smaller tiles.



We see $\frac{6}{8}$ is equivalent to $\frac{3}{4}$. Taking half of the six $\frac{1}{8}$ tiles gives us three $\frac{1}{8}$ tiles, which is $\frac{3}{8}$.

Therefore,

Equation:

$$\frac{1}{2} \cdot \frac{3}{4} = \frac{3}{8}$$

Example:

Multiplying Fractions

Exercise:

Problem: Use a diagram to model $\frac{1}{2} \cdot \frac{3}{4}$.

Solution:

Solution

First shade in $\frac{3}{4}$ of the rectangle.

We will take $\frac{1}{2}$ of this $\frac{3}{4}$, so we heavily shade $\frac{1}{2}$ of the shaded region.

Notice that 3 out of the 8 pieces are heavily shaded. This means that $\frac{3}{8}$ of the rectangle is heavily shaded.

Therefore, $\frac{1}{2}$ of $\frac{3}{4}$ is $\frac{3}{8}$, or $\frac{1}{2} \cdot \frac{3}{4} = \frac{3}{8}$.

Look at the result we got from the model in [link]. We found that $\frac{1}{2} \cdot \frac{3}{4} = \frac{3}{8}$. Do you notice that we could have gotten the same answer by multiplying the numerators and multiplying the denominators?

	$\frac{1}{2} \cdot \frac{3}{4}$
Multiply the numerators, and multiply the denominators.	$\frac{1}{2} \cdot \frac{3}{4}$
Simplify.	$\frac{3}{8}$

This leads to the definition of fraction multiplication. To multiply fractions, we multiply the numerators and multiply the denominators. Then we write the fraction in simplified form.

Note:
If a, b, c, and d are numbers where $b \neq 0$ and $d \neq 0$, then
Equation:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

Example:
Multiplying Fractions
Exercise:

Problem: Multiply, and write the answer in simplified form: $\frac{3}{4} \cdot \frac{1}{5}$.

Solution:
Solution

	$\frac{3}{4} \cdot \frac{1}{5}$
Multiply the numerators; multiply the denominators.	$\frac{3 \cdot 1}{4 \cdot 5}$
Simplify.	$\frac{3}{20}$

There are no common factors, so the fraction is simplified.

When multiplying fractions, the properties of positive and negative numbers still apply. It is a good idea to determine the sign of the product as the first step. In Example 4.26 we will multiply two negatives, so the product will be positive.

Example:
Multiplying Fractions with Negatives
Exercise:

Problem:

Multiply, and write the answer in simplified form: $-\frac{5}{8}\left(-\frac{2}{3}\right)$.

Solution:
Solution

	$-\frac{5}{8}\left(-\frac{2}{3}\right)$
The signs are the same, so the product is positive. Multiply the numerators, multiply the denominators.	$\frac{5\cdot2}{8\cdot3}$
Simplify.	$\frac{10}{24}$
Look for common factors in the numerator and denominator. Rewrite showing common factors.	$\frac{5\cdot\cancel{2}}{12\cdot\cancel{2}}$
Remove common factors.	$\frac{5}{12}$

Another way to find this product involves removing common factors earlier.

	$-\frac{5}{8}\left(-\frac{2}{3}\right)$
Determine the sign of the product. Multiply.	$\frac{5 \cdot 2}{8 \cdot 3}$
Show common factors and then remove them.	$\frac{5 \cdot \cancel{2}}{4 \cdot \cancel{2} \cdot 3}$
Multiply remaining factors.	$\frac{5}{12}$

We get the same result.

Example:
Multiplying Fractions with Negatives
Exercise:

Problem:

Multiply, and write the answer in simplified form: $-\frac{14}{15} \cdot \frac{20}{21}$.

Solution:
Solution

	$-\frac{14}{15} \cdot \frac{20}{21}$
Determine the sign of the product; multiply.	$-\frac{14}{15} \cdot \frac{20}{21}$
Are there any common factors in the numerator	

and the denominator? We know that 7 is a factor of 14 and 21, and 5 is a factor of 20 and 15.	
Rewrite showing common factors.	$-\frac{2\cdot\cancel{7}\cdot 4\cdot\cancel{5}}{3\cdot\cancel{5}\cdot 3\cdot\cancel{7}}$
Remove the common factors.	$-\frac{2\cdot 4}{3\cdot 3}$
Multiply the remaining factors.	$-\frac{8}{9}$

When multiplying a fraction by an integer, it may be helpful to write the integer as a fraction. Any integer, a, can be written as $\frac{a}{1}$. So, $3 = \frac{3}{1}$, for example.

Example:
Multiplying Fractions, Whole Numbers, and Terms
Exercise:

Problem: Multiply, and write the answer in simplified form:

ⓐ $\frac{1}{7}\cdot 56$

ⓑ $\frac{12}{5}(-20x)$

Solution:
Solution

ⓐ	
	$\frac{1}{7} \cdot 56$
Write 56 as a fraction.	$\frac{1}{7} \cdot \frac{56}{1}$
Determine the sign of the product; multiply.	$\frac{56}{7}$
Simplify.	8

ⓑ	
	$\frac{12}{5}(-20x)$
Write −20x as a fraction.	$\frac{12}{5}\left(\frac{-20x}{1}\right)$
Determine the sign of the product; multiply.	$-\frac{12 \cdot 20 \cdot x}{5 \cdot 1}$
Show common factors and then remove them.	$-\frac{12 \cdot 4 \cdot \cancel{5}x}{\cancel{5} \cdot 1}$
Multiply remaining factors; simplify.	−48x

Find Reciprocals [footnote]

Section material derived from Openstax Prealgebra: Fractions-Multiply and Divide Fractions

The fractions $\frac{2}{3}$ and $\frac{3}{2}$ are related to each other in a special way. So are $-\frac{10}{7}$ and $-\frac{7}{10}$. Do you see how? Besides looking like upside-down versions of one another, if we were to multiply these pairs of fractions, the product would be 1.

Equation:

$$\frac{2}{3} \cdot \frac{3}{2} = 1 \quad \text{and} \quad -\frac{10}{7}\left(-\frac{7}{10}\right) = 1$$

Such pairs of numbers are called reciprocals.

Note:
The **reciprocal** of the fraction $\frac{a}{b}$ is $\frac{b}{a}$, where $a \neq 0$ and $b \neq 0$,
A number and its reciprocal have a product of 1.
Equation:

$$\frac{a}{b} \cdot \frac{b}{a} = 1$$

To find the reciprocal of a fraction, we invert the fraction. This means that we place the numerator in the denominator and the denominator in the numerator.

To get a positive result when multiplying two numbers, the numbers must have the same sign. So reciprocals must have the same sign.

$$\frac{a}{b} \cdot \frac{b}{a} = 1 \text{ positive}$$

$$3 \cdot \frac{1}{3} = 1 \quad \text{and} \quad -3 \cdot \left(-\frac{1}{3}\right) = 1$$

both positive both negative

To find the reciprocal, keep the same sign and invert the fraction. The number zero does not have a reciprocal. Why? A number and its reciprocal multiply to 1. Is there any number r so that $0 \cdot r = 1$? No. So, the number 0 does not have a reciprocal.

Example:
Finding Reciprocals
Exercise:

Problem:

Find the reciprocal of each number. Then check that the product of each number and its reciprocal is 1.

ⓐ $\frac{4}{9}$
ⓑ $-\frac{1}{6}$
ⓒ $-\frac{14}{5}$
ⓓ 7

Solution:
Solution

To find the reciprocals, we keep the sign and invert the fractions.

ⓐ	
Find the reciprocal of $\frac{4}{9}$.	The reciprocal of $\frac{4}{9}$ is $\frac{9}{4}$.
Check:	
Multiply the number and its reciprocal.	$\frac{4}{9} \cdot \frac{9}{4}$
Multiply numerators and denominators.	$\frac{36}{36}$
Simplify.	$1\checkmark$

ⓑ	
Find the reciprocal of $-\frac{1}{6}$.	$-\frac{6}{1}$
Simplify.	-6
Check:	$-\frac{1}{6} \cdot (-6)$
	$1\checkmark$

ⓒ	

Find the reciprocal of $-\frac{14}{5}$.	$-\frac{5}{14}$
Check:	$-\frac{14}{5} \cdot \left(-\frac{5}{14}\right)$
	$\frac{70}{70}$
	$1\checkmark$

ⓓ	
Find the reciprocal of 7.	
Write 7 as a fraction.	$\frac{7}{1}$
Write the reciprocal of $\frac{7}{1}$.	$\frac{1}{7}$
Check:	$7 \cdot \left(\frac{1}{7}\right)$
	$1\checkmark$

In a previous chapter, we worked with opposites and absolute values. [link] compares opposites, absolute values, and reciprocals.

Opposite	Absolute Value	Reciprocal
has opposite sign	is never negative	has same sign, fraction inverts

Example:
Reciprocals and Opposites
Exercise:

Problem: Fill in the chart for each fraction in the left column:

Number	Opposite	Absolute Value	Reciprocal
$-\frac{3}{8}$			
$\frac{1}{2}$			
$\frac{9}{5}$			
-5			

Solution:
Solution

To find the opposite, change the sign. To find the absolute value, leave the positive numbers the same, but take the opposite of the negative

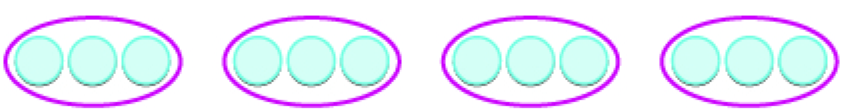
numbers. To find the reciprocal, keep the sign the same and invert the fraction.

Number	Opposite	Absolute Value	Reciprocal
$-\frac{3}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$-\frac{8}{3}$
$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	2
$\frac{9}{5}$	$-\frac{9}{5}$	$\frac{9}{5}$	$\frac{5}{9}$
-5	5	5	$-\frac{1}{5}$

Divide Fractions [footnote]

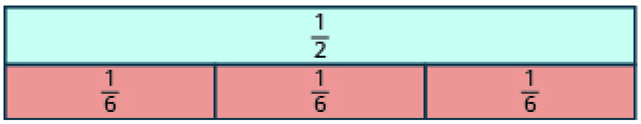
Section material derived from Openstax Prealgebra: Fractions-Multiply and Divide Fractions

Why is $12 \div 3 = 4$? We previously modeled this with counters. How many groups of 3 counters can be made from a group of 12 counters?



There are 4 groups of 3 counters. In other words, there are four 3s in 12. So, $12 \div 3 = 4$.

What about dividing fractions? Suppose we want to find the quotient: $\frac{1}{2} \div \frac{1}{6}$. We need to figure out how many $\frac{1}{6}$s there are in $\frac{1}{2}$. We can use fraction tiles to model this division. We start by lining up the half and sixth fraction tiles as shown in [link]. Notice, there are three $\frac{1}{6}$ tiles in $\frac{1}{2}$, so $\frac{1}{2} \div \frac{1}{6} = 3$.

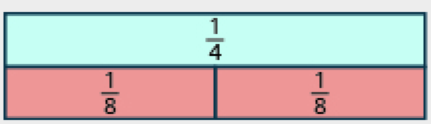


Example:
Dividing Fractions
Exercise:

Problem: Model: $\frac{1}{4} \div \frac{1}{8}$.

Solution:
Solution

We want to determine how many $\frac{1}{8}$s are in $\frac{1}{4}$. Start with one $\frac{1}{4}$ tile. Line up $\frac{1}{8}$ tiles underneath the $\frac{1}{4}$ tile.



There are two $\frac{1}{8}$s in $\frac{1}{4}$.

So, $\frac{1}{4} \div \frac{1}{8} = 2$.

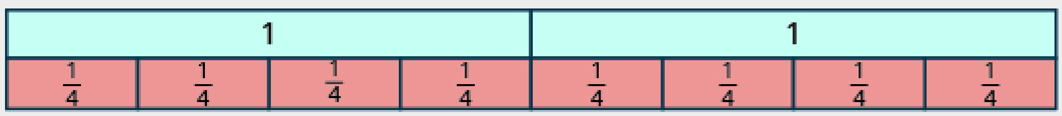
Example:

Dividing Fractions
Exercise:

Problem: Model: $2 \div \frac{1}{4}$.

Solution:
Solution

We are trying to determine how many $\frac{1}{4}$s there are in 2. We can model this as shown.



Because there are eight $\frac{1}{4}$s in 2, $2 \div \frac{1}{4} = 8$.

Let's use money to model $2 \div \frac{1}{4}$ in another way. We often read $\frac{1}{4}$ as a 'quarter', and we know that a quarter is one-fourth of a dollar as shown in [link]. So we can think of $2 \div \frac{1}{4}$ as, "How many quarters are there in two dollars?" One dollar is 4 quarters, so 2 dollars would be 8 quarters. So again, $2 \div \frac{1}{4} = 8$.

The U.S. coin called a quarte

r is
worth
one-
fourth
of a
dollar.

Using fraction tiles, we showed that $\frac{1}{2} \div \frac{1}{6} = 3$. Notice that $\frac{1}{2} \cdot \frac{6}{1} = 3$ also. How are $\frac{1}{6}$ and $\frac{6}{1}$ related? They are reciprocals. This leads us to the procedure for fraction division.

Note:
If $a, b, c,$ and d are numbers where $b \neq 0, c \neq 0,$ and $d \neq 0$, then
Equation:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$$

To divide fractions, multiply the first fraction by the reciprocal of the second.

We need to say $b \neq 0, c \neq 0$ and $d \neq 0$ to be sure we don't divide by zero.

Example:
Dividing Fractions
Exercise:

Problem:

Divide, and write the answer in simplified form: $\frac{2}{5} \div \left(-\frac{3}{7}\right)$.

Solution:
Solution

	$\frac{2}{5} \div \left(-\frac{3}{7}\right)$
Multiply the first fraction by the reciprocal of the second.	$\frac{2}{5}\left(-\frac{7}{3}\right)$
Multiply. The product is negative.	$-\frac{14}{15}$

Example:
Dividing Fractions
Exercise:

Problem: Divide, and write the answer in simplified form: $\frac{2}{3} \div \frac{n}{5}$.

Solution:
Solution

	$\frac{2}{3} \div \frac{n}{5}$
Multiply the first fraction by the reciprocal of the	$\frac{2}{3} \cdot \frac{5}{n}$

second.	
Multiply.	$\frac{10}{3n}$

Example:
Dividing Fractions
Exercise:

Problem:

Divide, and write the answer in simplified form: $-\frac{3}{4} \div \left(-\frac{7}{8}\right)$.

Solution:
Solution

	$-\frac{3}{4} \div \left(-\frac{7}{8}\right)$
Multiply the first fraction by the reciprocal of the second.	$-\frac{3}{4} \cdot \left(-\frac{8}{7}\right)$
Multiply. Remember to determine the sign first.	$\frac{3 \cdot 8}{4 \cdot 7}$
Rewrite to show common factors.	$\frac{3 \cdot \cancel{4} \cdot 2}{\cancel{4} \cdot 7}$
Remove common factors and simplify.	$\frac{6}{7}$

Example:
Dividing Fractions
Exercise:

Problem: Divide, and write the answer in simplified form: $\frac{7}{18} \div \frac{14}{27}$.

Solution:
Solution

	$\frac{7}{18} \div \frac{14}{27}$
Multiply the first fraction by the reciprocal of the second.	$\frac{7}{18} \cdot \frac{27}{14}$
Multiply.	$\frac{7 \cdot 27}{18 \cdot 14}$
Rewrite showing common factors.	$\frac{\cancel{7} \cdot \cancel{9} \cdot 3}{\cancel{9} \cdot 2 \cdot \cancel{7} \cdot 2}$
Remove common factors.	$\frac{3}{2 \cdot 2}$
Simplify.	$\frac{3}{4}$

Note:
ACCESS ADDITIONAL ONLINE RESOURCES

Summary of Fraction Operations [footnote]

Section material derived from Openstx Prealgebra: Fractions-Add and Subtract Fractions with Different Denominators

Fraction multiplication: Multiply the numerators and multiply the denominators.
Equation:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

Fraction division: Multiply the first fraction by the reciprocal of the second.
Equation:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$$

Fraction addition: Add the numerators and place the sum over the common denominator. If the fractions have different denominators, first convert them to equivalent forms with the LCD.
Equation:

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$$

Fraction subtraction: Subtract the numerators and place the difference over the common denominator. If the fractions have different denominators, first convert them to equivalent forms with the LCD.
Equation:

$$\frac{a}{c} - \frac{b}{c} = \frac{a - b}{c}$$

Convert Fractions to Decimals [footnote]

Section material derived from Openstax Prealgebra: Decimals-Decimals and Fractions

Remember that the fraction bar indicates division. So $\frac{4}{5}$ can be written as $4 \div 5$ or $5\overline{)4}$. This means that we can convert a fraction to a decimal by treating it as a division problem.

Note:
To convert a fraction to a decimal, divide the numerator of the fraction by the denominator of the fraction.

Example:
Convert a Fraction to a Decimal
Exercise:

Problem: Write the fraction $\frac{3}{4}$ as a decimal.

Solution:
Solution

A fraction bar means division, so we can write the fraction $\frac{3}{4}$ using division.	$4\overline{)3}$
Divide.	$\begin{array}{r} 0.75 \\ 4\overline{)3.00} \\ \underline{28} \\ 20 \\ \underline{20} \\ 0 \end{array}$
	So the fraction $\frac{3}{4}$ is equal to 0.75.

Key Concepts

- **Simplify a fraction.**

  Rewrite the numerator and denominator to show the common factors.
  If needed, factor the numerator and denominator into prime numbers.
  Simplify, using the equivalent fractions property, by removing common factors.
  Multiply any remaining factors.

- **Fraction Addition**
  - If a, b, and c are numbers where $c \neq 0$, then $\frac{a}{c} + \frac{b}{c} = \frac{a+c}{c}$.
  - To add fractions, add the numerators and place the sum over the common denominator.

- **Fraction Subtraction**

    - If a, b, and c are numbers where $c \neq 0$, then $\frac{a}{c} - \frac{b}{c} = \frac{a-c}{c}$.
    - To subtract fractions, subtract the numerators and place the difference over the common denominator.

- **Find the least common denominator (LCD) of two fractions.**

  Factor each denominator into its primes.
  List the primes, matching primes in columns when possible.
  Bring down the columns.
  Multiply the factors. The product is the LCM of the denominators.
  The LCM of the denominators is the LCD of the fractions.

- **Equivalent Fractions Property**

    - If a, b, and c are whole numbers where $b \neq 0$, $c \neq 0$ then $\frac{a}{b} = \frac{a \cdot c}{b \cdot c}$ and $\frac{a \cdot c}{b \cdot c} = \frac{a}{b}$

- **Convert two fractions to equivalent fractions with their LCD as the common denominator.**

  Find the LCD.
  For each fraction, determine the number needed to multiply the denominator to get the LCD.
  Use the Equivalent Fractions Property to multiply the numerator and denominator by the number from Step 2.
  Simplify the numerator and denominator.

- **Add or subtract fractions with different denominators.**

  Find the LCD.
  Convert each fraction to an equivalent form with the LCD as the denominator.
  Add or subtract the fractions.
  Write the result in simplified form.

- **Summary of Fraction Operations**

- **Fraction multiplication:** Multiply the numerators and multiply the denominators.
  Equation:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

- **Fraction division:** Multiply the first fraction by the reciprocal of the second.
  Equation:

$$\frac{a}{b} + \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$$

- **Fraction addition:** Add the numerators and place the sum over the common denominator. If the fractions have different denominators, first convert them to equivalent forms with the LCD.
  Equation:

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$$

- **Fraction subtraction:** Subtract the numerators and place the difference over the common denominator. If the fractions have different denominators, first convert them to equivalent forms with the LCD.
  Equation:

$$\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$$

- **Simplify complex fractions.**

  Simplify the numerator.
  Simplify the denominator.
  Divide the numerator by the denominator.
  Simplify if possible.

- **Reciprocal**
  - A number and its reciprocal have a product of 1. $\frac{a}{b} \cdot \frac{b}{a} = 1$
  -
Opposite	**Absolute Value**	**Reciprocal**
has opposite sign	is never negative	has same sign, fraction inverts

- **Convert a Fraction to a Decimal:** to convert a fraction to a decimal, divide the numerator of the fraction by the denominator of the fraction.
- **Add mixed numbers with a common denominator.**

  Add the whole numbers.
  Add the fractions.
  Simplify, if possible.

- **Subtract mixed numbers with common denominators.**

  Rewrite the problem in vertical form.

Compare the two fractions.	If the top fraction is larger than the bottom fraction, go to Step 3.	If not, in the top mixed number, take one whole and add it to the fraction part, making a mixed number with an improper fraction.

  Subtract the fractions.
  Subtract the whole numbers.
  Simplify, if possible.

- **Subtract mixed numbers with common denominators as improper fractions.**

Rewrite the mixed numbers as improper fractions.
Subtract the numerators.
Write the answer as a mixed number, simplifying the fraction part, if possible.

Glossary

Fraction
: A fraction is written $\frac{a}{b}$. in a fraction, a is the numerator and b is the denominator. A fraction represents parts of a whole. The denominator b is the number of equal parts the whole has been divided into, and the numerator a indicates how many parts are included.

Least Common Denominator (LCD)
: The least common denominator (LCD) of two fractions is the least common multiple (LCM) of their denominators.

Mixed Number
: A mixed number consists of a whole number a and a fraction $\frac{b}{c}$ where $c \neq 0$. It is written as $a\frac{b}{c}$, where $c \neq 0$.

Proper and Improper Fractions
: The fraction $\frac{a}{b}$ is *proper* if $a < b$and *improper* if $a > b$.

Reciprocal
: The reciprocal of the fraction $\frac{a}{b}$ is $\frac{b}{a}$ where $a \neq 0$ and $b \neq 0$.

Simplified Fraction
: A fraction is considered simplified if there are no common factors in the numerator and denominator.

Probabilities: Lessons 4.A - 4.B
By the end of this section, you will be able to:

- Use the definition of percent
- Convert percents to fractions and decimals
- Ratios with decimals
- Use Probability with and/or events

Note:
In case you missed something in class, or just want to review a specific topic covered in this Module, here is a list of topics covered:

1. Write Ratios as Fractions [link]
2. Ratios involving Decimals [link]
3. Conversions: Relationship between Fraction, Decimals, and Percents [link]
4. Definition of Probability [link]
5. Independent Events [link]
6. "And" and "or" Events [link]
7. Conditional Probabilities [link]
8. Key Concepts [link]

Write a Ratio as a Fraction [footnote]

Section material derived from Openstax Prealgebra: Decimals-Ratios and Rate

When you apply for a mortgage, the loan officer will compare your total debt to your total income to decide if you qualify for the loan. This comparison is called the debt-to-income ratio. A **ratio** compares two quantities that are measured with the same unit. If we compare a and b, the ratio is written as a to b, $\frac{a}{b}$, or $a{:}b$.

Note:

A **ratio** compares two numbers or two quantities that are measured with the same unit. The ratio of a to b is written a to b, $\frac{a}{b}$, or $a{:}b$.

In this section, we will use the fraction notation. When a ratio is written in fraction form, the fraction should be simplified. If it is an improper fraction, we do not change it to a mixed number. Because a ratio compares two quantities, we would leave a ratio as $\frac{4}{1}$ instead of simplifying it to 4 so that we can see the two parts of the ratio.

Example:
Writing Ratios as Fractions
Exercise:

Problem: Write each ratio as a fraction: ⓐ 15 to 27 ⓑ 45 to 18.

Solution:
Solution

ⓐ	
	$15 \text{ to } 27$
Write as a fraction with the first number in the numerator and the second in the denominator.	$\frac{15}{27}$
Simplify the fraction.	$\frac{5}{9}$

ⓑ	
	45 to 18
Write as a fraction with the first number in the numerator and the second in the denominator.	$\frac{45}{18}$
Simplify.	$\frac{5}{2}$

We leave the ratio in ⓑ as an improper fraction.

Ratios Involving Decimals [footnote]

Section material derived from Openstax Prealgebra: Decimals-Ratios and Rate

We will often work with ratios of decimals, especially when we have ratios involving money. In these cases, we can eliminate the decimals by using the Equivalent Fractions Property to convert the ratio to a fraction with whole numbers in the numerator and denominator.

For example, consider the ratio 0.8 to 0.05. We can write it as a fraction with decimals and then multiply the numerator and denominator by 100 to eliminate the decimals.

$$\frac{0.8}{0.05}$$

$$\frac{(0.8)100}{(0.05)100}$$

$$\frac{80}{5}$$

Do you see a shortcut to find the equivalent fraction? Notice that $0.8 = \frac{8}{10}$ and $0.05 = \frac{5}{100}$. The least common denominator of $\frac{8}{10}$ and $\frac{5}{100}$ is 100. By multiplying the numerator and denominator of $\frac{0.8}{0.05}$ by 100, we 'moved' the decimal two places to the right to get the equivalent fraction with no decimals.

Now that we understand the math behind the process, we can find the fraction with no decimals like this:

	$\frac{0.80}{0.05}$
"Move" the decimal 2 places.	$\frac{80}{5}$
Simplify.	$\frac{16}{1}$

You do not have to write out every step when you multiply the numerator and denominator by powers of ten. As long as you move both decimal places the same number of places, the ratio will remain the same.

Example:
Writing Ratios as Fractions
Exercise:

Problem: Write each ratio as a fraction of whole numbers:

ⓐ 4.8 to 11.2

ⓑ 2.7 to 0.54

Solution:
Solution

ⓐ 4.8 to 11.2	
Write as a fraction.	$\frac{4.8}{11.2}$
Rewrite as an equivalent fraction without decimals, by moving both decimal points 1 place to the right.	$\frac{48}{112}$
Simplify.	$\frac{3}{7}$

So 4.8 to 11.2 is equivalent to $\frac{3}{7}$.

ⓑ The numerator has one decimal place and the denominator has 2. To clear both decimals we need to move the decimal 2 places to the right. 2.7 to 0.54	
Write as a fraction.	$\frac{2.7}{0.54}$
Move both decimals right two places.	$\frac{270}{54}$
Simplify.	$\frac{5}{1}$

So 2.7 to 0.54 is equivalent to $\frac{5}{1}$.

The Relationship Between Fractions, Decimals, and Percents – Making Conversions [footnote]

Section material derived from Openstax Prealgebra: Decimals-Ratios and Rate

Since a percent is a ratio, and a ratio can be written as a fraction, and a fraction can be written as a decimal, any of these forms can be converted to any other.

Before we proceed to the next section, let's review the conversion techniques learned in *Previous Units* .

To Convert a Fraction	**To Convert a Decimal**	**To Convert a Percent**
To a decimal: Divide the numerator by the denominator	To a fraction: Read the decimal and reduce the resulting fraction	To a decimal: Move the decimal point 2 places to the left and drop the % symbol
To a percent: Convert the fraction first to a decimal, then move the decimal point 2 places to the right and affix the % symbol.	To a percent: Move the decimal point 2 places to the right and affix the % symbol	To a fraction: Drop the % sign and write the number “over” 100. Reduce, if possible.

Conversion Techniques – Fractions, Decimals, Percents

Example: Convert Percent to Decimal
Convert 12% to a decimal.
$12\% = \frac{12}{100} = 0.12$
Note that

$$12\% = 12.\% = 0.12$$

The % symbol is dropped, and the decimal point moves 2 places to the left.

Example: Convert a Decimal to Percent
Convert 0.75 to a percent.
$0.75 = \frac{75}{100} = 75\%$
Note that

$$0.75 = 75\% = 75.\%$$

The % symbol is affixed, and the decimal point moves 2 units to the right.

Example: Convert a Fraction to Percent
Convert $\frac{3}{5}$ to a percent.
We see in [link] that we can convert a decimal to a percent. We also know that we can convert a fraction to a decimal. Thus, we can see that if we first convert the fraction to a decimal, we can then convert the decimal to a percent.

$$\frac{3}{5} \rightarrow \begin{array}{r} .6 \\ 5\overline{)3.0} \\ \underline{3\ 0} \\ 0 \end{array} \text{ or } \frac{3}{5} = 0.6 = \frac{6}{10} = \frac{60}{100} = 60\%$$

Example: Convert Percent to a Fraction
Convert 42% to a fraction.
$42\% = \frac{42}{100} = \frac{21}{50}$
or
$42\% = 0.42 = \frac{42}{100} = \frac{21}{50}$

The Defintion of Probability [footnote]

Section material derived from Openstax Prealgebra: Decimals-Averages and Probability

Probability is a mathematical tool used to study randomness. It deals with the chance (the likelihood) of an event occurring. For example, if you toss a **fair**

coin four times, the outcomes may not be two heads and two tails. However, if you toss the same coin 4,000 times, the outcomes will be close to half heads and half tails. The expected theoretical probability of heads in any one toss is $\frac{1}{2}$ or 0.5. Even though the outcomes of a few repetitions are uncertain, there is a regular pattern of outcomes when there are many repetitions. After reading about the English statistician Karl **Pearson** who tossed a coin 24,000 times with a result of 12,012 heads, one of the authors tossed a coin 2,000 times. The results were 996 heads. The fraction $\frac{996}{2000}$ is equal to 0.498 which is very close to 0.5, the expected probability.

The probability of an event tells us how likely that event is to occur. We usually write probabilities as fractions or decimals.

An **event** is any combination of outcomes. Upper case letters like *A* and *B* represent events. For example, if the experiment is to flip one fair coin, event *A* might be getting at most one head. The probability of an event *A* is written $P(A)$.

For example, picture a fruit bowl that contains five pieces of fruit - three bananas and two apples.

If you want to choose one piece of fruit to eat for a snack and don't care what it is, there is a $\frac{3}{5}$ probability you will choose a banana, because there are three bananas out of the total of five pieces of fruit. The probability of an event is the number of favorable outcomes divided by the total number of outcomes.

$$\text{Probability of an event} = \frac{\text{number of favorable outcomes}}{\text{total number of outcomes}}$$

$$\text{Probability of choosing a banana} = \frac{3}{5}$$

There are 3 bananas.
There are 5 pieces of fruit.

Note:
The **probability** of an event is the number of favorable outcomes divided by the total number of outcomes possible.
Equation:

$$\text{Probability} = \frac{\text{number of favorable outcomes}}{\text{total number of outcomes}}$$

Converting the fraction $\frac{3}{5}$ to a decimal, we would say there is a 0.6 probability of choosing a banana.

Equation:

$$\begin{aligned}\text{Probability of choosing a banana} &= \frac{3}{5} \\ \text{Probability of choosing a banana} &= 0.6\end{aligned}$$

This basic definition of probability assumes that all the outcomes are equally likely to occur. If you study probabilities in a later math class, you'll learn about several other ways to calculate probabilities.

Example:
Determining Probability
Exercise:

Problem:

The ski club is holding a raffle to raise money. They sold 100 tickets. All of the tickets are placed in a jar. One ticket will be pulled out of the jar at random, and the winner will receive a prize. Cherie bought one raffle ticket.

ⓐ Find the probability she will win the prize.

ⓑ Convert the fraction to a decimal.

Solution:
Solution

ⓐ	
What are	The probability Cherie wins the prize.

you asked to find?	
What is the number of favorable outcomes?	1, because Cherie has 1 ticket.
Use the definition of probability.	$\text{Probability of an event} = \frac{\text{number of favorable outcomes}}{\text{total number of outcomes}}$
Substitute into the numerator and denominator.	$\text{Probability Cherie wins} = \frac{1}{100}$

ⓑ	
Convert the fraction to a decimal.	
Write the probability as a fraction.	$\text{Probability} = \frac{1}{100}$
Convert the fraction to a decimal.	$\text{Probability} = 0.01$

Example:
Determining Probability
Exercise:

Problem:

Three women and five men interviewed for a job. One of the candidates will be offered the job.

ⓐ Find the probability the job is offered to a woman.

ⓑ Convert the fraction to a decimal.

Solution:
Solution

ⓐ	
What are you asked to find?	The probability the job is offered to a woman.
What is the number of favorable outcomes?	3, because there are three women.
What are the total number of outcomes?	8, because 8 people interviewed.
Use the definition of probability.	$\text{Probability of an event} = \frac{\text{number of favorable outcomes}}{\text{total number of outcomes}}$
Substitute into the	$\text{Probability} = \frac{3}{8}$

numerator and denominator.

ⓑ	
Convert the fraction to a decimal.	
Write the probability as a fraction.	$\text{Probability} = \frac{3}{8}$
Convert the fraction to a decimal.	$\text{Probability} = 0.375$

Independent Events [footnote]

Section material derived from Openstax Introductory Statistics: Probability Topics-Independent and Mutually Exclusive Events

Two events are independent if the following are true:

- $P(A|B) = P(A)$
- $P(B|A) = P(B)$
- $P(A \text{ AND } B) = P(A)P(B)$

Two events A and B are **independent** if the knowledge that one occurred does not affect the chance the other occurs. For example, the outcomes of two roles of a fair die are independent events. The outcome of the first roll does not change the probability for the outcome of the second roll. To show two events are independent, you must show **only one** of the above conditions. If two events are NOT independent, then we say that they are **dependent**.

Sampling may be done **with replacement** or **without replacement**.

- **With replacement**: If each member of a population is replaced after it is picked, then that member has the possibility of being chosen more than once. When sampling is done with replacement, then events are considered to be independent, meaning the result of the first pick will not change the probabilities for the second pick.
- **Without replacement**: When sampling is done without replacement, each member of a population may be chosen only once. In this case, the probabilities for the second pick are affected by the result of the first pick. The events are considered to be dependent or not independent.

If it is not known whether A and B are independent or dependent, **assume they are dependent until you can show otherwise**.

Example:
Probability With and Without Replacement

You have a fair, well-shuffled deck of 52 cards. It consists of four suits. The suits are clubs, diamonds, hearts and spades. There are 13 cards in each suit consisting of 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, J (jack), Q (queen), K (king) of that suit.

a. Sampling with replacement:

Suppose you pick three cards with replacement. The first card you pick out of the 52 cards is the Q of spades. You put this card back, reshuffle the cards and pick a second card from the 52-card deck. It is the ten of clubs. You put this card back, reshuffle the cards and pick a third card from the 52-card deck. This time, the card is the Q of spades again. Your picks are {Q of spades, ten of clubs, Q of spades}. You have picked the Q of spades twice. You pick each card from the 52-card deck.

b. Sampling without replacement:

Suppose you pick three cards without replacement. The first card you pick out of the 52 cards is the K of hearts. You put this card aside and pick the second card from the 51 cards remaining in the deck. It is the three of diamonds. You put this card aside and pick the third card from the remaining 50 cards in the deck. The third card is the J of spades. Your picks are {K of hearts, three of diamonds, J of spades}. Because you have picked the cards without replacement, you cannot pick the same card twice.

Example:
Probability With and Without Replacement
Exercise:

Problem:

You have a fair, well-shuffled deck of 52 cards. It consists of four suits. The suits are clubs, diamonds, hearts, and spades. There are 13 cards in each suit consisting of 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, J (jack), Q (queen), and K (king) of that suit. S = spades, H = Hearts, D = Diamonds, C = Clubs.

a. Suppose you pick four cards, but do not put any cards back into the deck. Your cards are QS, 1D, 1C, QD.
b. Suppose you pick four cards and put each card back before you pick the next card. Your cards are KH, 7D, 6D, KH.

Which of a. or b. did you sample with replacement and which did you sample without replacement?

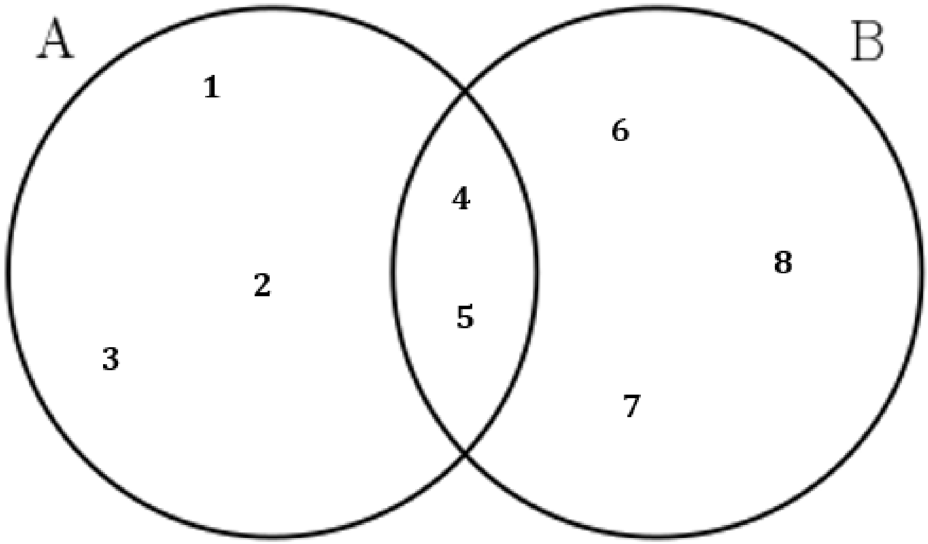
Solution:

a. Without replacement; b. With replacement

"AND" and "OR" Events [footnote]

Section material derived from Openstax Introductory Statistics: Probability Topics-Terminology

"AND" Event:
An outcome is in the event A AND B if the outcome is in both A and B at the same time. For example, let A and B be {1, 2, 3, 4, 5} and {4, 5, 6, 7, 8}, respectively. Then A AND B = {4, 5}. This can be shown by creating a venn diagram to list all the elements of set A and the elements of set B.



Considering both sets together, there are a total of ten elements possible. If only two of the elements are within both sets, then that means that the probability of finding a value in both set A and B,

Equation:

$$P\left(A \text{ AND } B\right) = \frac{2}{8} = \frac{1}{4}$$

However, when dealing with ratios as probabilities, we can represent the "AND" probability differently. Consider flipping a coin and rolling a six-sided die. You can either flip a head, or a tail on the coin, and you can roll any number between one and six on the die. These possibilities can be represented in a table of outcomes.

	1	**2**	**3**	**4**	**5**	**6**
Heads	H, 1	H, 2	H, 3	H, 4	H, 5	H, 6
Tails	T, 1	T, 2	T, 3	T, 4	T, 5	T, 6

Notice that all of the possible outcomes are listed for both flipping the coin and rolling the die. This gives us total possible outcomes of 12; which will be our whole, or denominator, in our probability ratios. If we wanted to find the probability of flipping a tail and rolling a 3, we would need to identify each situation where each of these occurrences appears in the table. First, let's count the number of times flipping a tail on the coin appears in the table.

	1	2	3	4	5	6
Heads	H, 1	H, 2	H, 3	H, 4	H, 5	H, 6
Tails	**T**, 1	**T**, 2	**T**, 3	**T**, 4	**T**, 5	**T**, 6

Equation:

$$\mathrm{P}\left(\text{Flipping Tails}\right) = \frac{\text{Times a tails is flipped}}{\text{Total outcomes possible}} = \frac{6}{12} = \frac{1}{2}$$

What we have essentially done above is break a whole of 12 possible outcomes into two groups of 6 known outcomes (flipping tails) which is the same as taking half of the original total outcomes. Now I am going to want to take the new group of 6 possibles (all the times tails can be flipped) and take out only the times a three was rolled with the tail.

	1	2	3	4	5	6
Heads	H, 1	H, 2	H, 3	H, 4	H, 5	H, 6
Tails	T, 1	T, 2	**T, 3**	T, 4	T, 5	T, 6

Equation:

$$\mathrm{P}\left(\text{Rolling a 3}\right) = \frac{\text{Times a 3 is rolled}}{\text{Total possible outcomes after tail has flipped}} = \frac{1}{6}$$

Let's think about the probabilities we have taken so far. We started by considering half of the initial total; the twelve possibilities for flipping coins and

rolling a die. We took a part of a whole. The answer we got from that was then used as the new whole for the second step. So, really, we took a fraction of a fraction, right? Here is how that looks mathematically:

Equation:

$$\mathrm{P}\left(\mathrm{T},3\right) = \frac{\text{Probability of flipping tails}}{6} = \frac{\frac{1}{2}}{6}$$

What do we do when dividing fractions? We flip and multiply! So the above equations can be rewritten as multiplication as follows:

Equation:

$$\mathrm{P}\left(\mathrm{T},3\right) = \frac{\text{Probability of flipping tails}}{6} = \frac{1}{2} * \frac{1}{6} = \frac{1}{12}$$

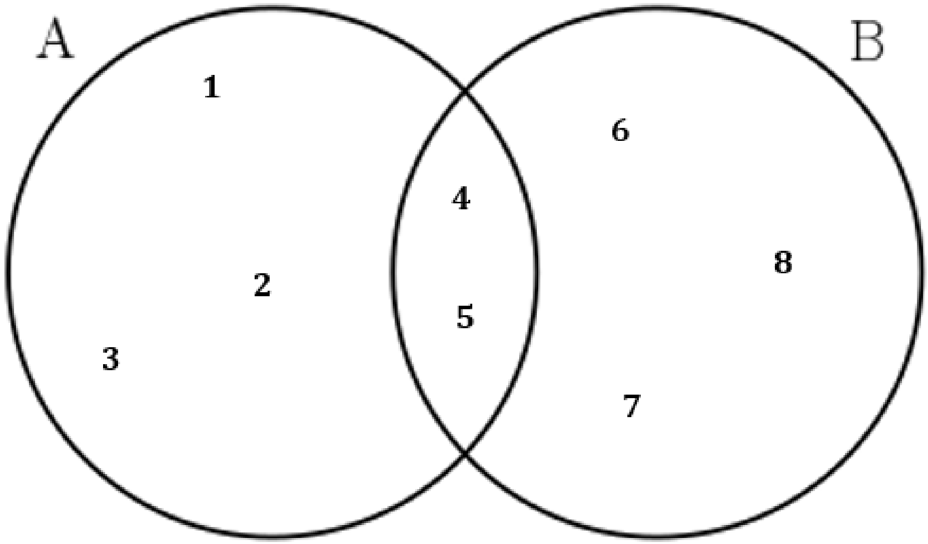
By multiplying the two separate probabilities, we calculated the same outcome as we would have gotten by simply counting the number of occurrences that include both tails AND a three in the same box. This is the general formula for calculating the probability of two independent events occurring together and can be used instead of having to draw out an outcome table every time.

$$P(A\ \boldsymbol{and}\ B) = P(A) \cdot P(B)$$

"OR" Event:

An outcome is in the event *A* OR *B* if the outcome is in *A* or is in *B* or is in both *A* and *B*. For example, let *A* = {1, 2, 3, 4, 5} and *B* = {4, 5, 6, 7, 8}. *A* OR *B* = {1, 2, 3, 4, 5, 6, 7, 8}. Notice that 4 and 5 are NOT listed twice.

We can look at the same Venn diagram from above again, from the OR perspective.



When looking at elements that are present in set A OR set B, that means any of the elements that qualify for either one of the two sets. This is why all the elements from both set A and set B would be in the union of A and B. This would be a probability of

Equation:

$$P\left(\text{A OR B}\right) = \frac{4}{8} + \frac{4}{8} = \frac{8}{8} = 1$$

When dealing with probabilities involving independent events again looks slightly different.

	1	2	3	4	5	6
Heads	H, 1	H, 2	H, 3	H, 4	H, 5	H, 6
Tails	T, 1	T, 2	T, 3	T, 4	T, 5	T, 6

Using the same example from above, let's say that this time we want either flipping a tail OR rolling a 3. We must first identify all the times that we flip a

tail in the table of outcomes.

	1	2	3	4	5	6
Heads	H, 1	H, 2	H, 3	H, 4	H, 5	H, 6
Tails	**T**, 1	**T**, 2	**T**, 3	**T**, 4	**T**, 5	**T**, 6

Next, we can highlight all the times a three is rolled on the die. Many students would say that there are eight different instances where one of the two possible outcomes occurred. But there is a problem. Do you see it?

	1	2	3	4	5	6
Heads	H, 1	H, 2	H**, 3**	H, 4	H, 5	H, 6
Tails	**T,** 1	**T,** 2	**T, 3**	**T,** 4	**T,** 5	**T,** 6

In counting the times a tail is flipped and a three is rolled separately, we have counted one of the possible occurrences twice. According to our table, there should be seven instances where a tail OR a 3 occurred, but if you're not careful, you might accidentally count twice where they BOTH happened. In order to prevent this from happening any time we calculate the OR probability, we have to make sure that we removed the AND calculation of the two occurrences.

$$P(A\ \boldsymbol{or}\ B) = P(A) + P(B) - P(A\ \boldsymbol{and}\ B)$$

Conditional Probability [footnote]

Section material derived from Openstax Introductory Statistics: Probability of Topics-Terminology

The **conditional probability** of *A* given *B* is written *P*(*A*|*B*). *P*(*A*|*B*) is the probability that event *A* will occur given that the event *B* has already occurred. **A conditional reduces the sample space**. We calculate the probability of *A* from the reduced sample space *B*. The formula to calculate *P*(*A*|*B*) is *P*(*A*|*B*) = $\frac{P(A \text{AND} B)}{P(B)}$ where *P*(*B*) is greater than zero.

Example:
Conditional Probability
For example, suppose we toss one fair, six-sided die. The sample space *S* = {1, 2, 3, 4, 5, 6}. Let *A* = face is 2 or 3 and *B* = face is even (2, 4, 6). To calculate *P*(*A*|*B*), we count the number of outcomes 2 or 3 in the sample space *B* = {2, 4, 6}. Then we divide that by the number of outcomes *B* (rather than *S*).
We get the same result by using the formula. Remember that *S* has six outcomes.

$$P(A|B) = \frac{P(A \text{ AND } B)}{P(B)} = \frac{\frac{\text{(the number of outcomes that are 2 or 3 and even in } S)}{6}}{\frac{\text{(the number of outcomes that are even in } S)}{6}} = \frac{\frac{1}{6}}{\frac{3}{6}} = \frac{1}{3}$$

Key Concepts

- A **ratio** compares two numbers or two quantities that are measured with the same unit. The ratio of a to b is written a to b, $\frac{a}{b}$, or a:b.
- **Conversion Rules**

To Convert a Fraction	To Convert a Decimal	To Convert a Percent
To a decimal: Divide the numerator by the denominator	To a fraction: Read the decimal and reduce the	To a decimal: Move the decimal point 2 places to the left

	resulting fraction	and drop the % symbol
To a percent: Convert the fraction first to a decimal, then move the decimal point 2 places to the right and affix the % symbol.	To a percent: Move the decimal point 2 places to the right and affix the % symbol	To a fraction: Drop the % sign and write the number "over" 100. Reduce, if possible.

- **The definition of a probability is** $\text{Probability} = \frac{\text{number of favorable outcomes}}{\text{total number of outcomes}}$
- **Independent events are defined as:**
  - $P(A|B) = P(A)$
  - $P(B|A) = P(B)$
  - $P(A \text{ AND } B) = P(A)P(B)$
- **Sampling can be conducted either**
  - with replacement, or
  - without replacement
- **AND events** $P(\text{A and B }) = P(A\ )*\ P(B)$
- **OR events** $P(\text{A or B }) = P(A\ ) + P(B\ )$ - $P(\text{A and B})$
- **Conditional Probability** $P(A|B) = \frac{P(A \text{ AND } B)}{P(B)}$

Glossary

AND Event
: An outcome is in the event A AND B if the outcome is in both A AND B at the same time.

Conditional Probability
: the likelihood that an event will occur given that another event has already occurred

Dependent Events

If two events are NOT independent, then we say that they are dependent.

Event

a subset of the set of all outcomes of an experiment; the set of all outcomes of an experiment is called a **sample space** and is usually denoted by *S*. An event is an arbitrary subset in *S*. It can contain one outcome, two outcomes, no outcomes (empty subset), the entire sample space, and the like. Standard notations for events are capital letters such as *A*, *B*, *C*, and so on.

Independent Events

Two events are considered independent if the knowledge that one occurred does not affect the chance the other occurs.

OR Event

An outcome is in the event A OR B if the outcome is in A or is in B or is in both A and B.

Percent

A perfecnt is a ratio whose denominator is 100.

Unit Rates and Conversions Lessons 5.A - 5.B
By the end of this section, you will be able to:

- Make unit conversions in the U.S. and metric system
- Use mixed units of measurement in the U.S. and metric system
- Convert between the U.S. and the metric systems of measurement
- Use weighted Averages

This modules works with the Corequistie MAT 1043 (QR) and NCBO course.

Note:
In case you missed something in class, or just want to review a specific topic covered in this Module, here is a list of topics covered:

1. Rate as a Fraction [link]
2. Ratio of Two Measurements in Different Units [link]
3. Find Unit Rates [link]
4. Unit Conversions in US [link]
5. Unit Conversions in Metric [link]
6. Convert Between US and Metric [link]
7. Key Concepts [link]

In this section we will see how to convert among different types of units, such as feet to miles or kilograms to pounds. The basic idea in all of the unit conversions will be to use a form of 1, the multiplicative identity, to change the units but not the value of a quantity.

Write a Rate as a Fraction [footnote]

Section material derived from Openstax Prealgebra: Decimals-Ratios and Rate

Frequently we want to compare two different types of measurements, such as miles to gallons. To make this comparison, we use a **rate**. Examples of rates are 120 miles in 2 hours, 160 words in 4 minutes, and $5 dollars per 64 ounces.

Note:
A **rate** compares two quantities of different units. A rate is usually written as a fraction.

When writing a fraction as a rate, we put the first given amount with its units in the numerator and the second amount with its units in the denominator. When rates are simplified, the units remain in the numerator and denominator.

Example:
Writing Rate as a Fraction
Exercise:

Problem: Bob drove his car 525 miles in 9 hours. Write this rate as a fraction.

Solution:
Solution

	525 miles in 9 hours
Write as a fraction, with 525 miles in the numerator and 9 hours in the denominator.	$\frac{525 \text{ miles}}{9 \text{ hours}}$
	$\frac{175 \text{ miles}}{3 \text{ hours}}$

So 525 miles in 9 hours is equivalent to $\frac{175 \text{ miles}}{3 \text{ hours}}$.

Ratios of Two Measurements in Different Units [footnote]

Section material derived from Openstax Prealgebra: Decimals-Ratios and Rate

To find the ratio of two measurements, we must make sure the quantities have been measured with the same unit. If the measurements are not in the same units, we must first convert them to the same units.

We know that to simplify a fraction, we divide out common factors. Similarly in a ratio of measurements, we divide out the common unit.

Example:
Writing Rate as a Ratio
Exercise:

Problem:

The Americans with Disabilities Act (ADA) Guidelines for wheel chair ramps require a maximum vertical rise of 1 inch for every 1 foot of horizontal run. What is the ratio of the rise to the run?

Solution:
Solution

In a ratio, the measurements must be in the same units. We can change feet to inches, or inches to feet. It is usually easier to convert to the smaller unit, since this avoids introducing more fractions into the problem.

Write the words that express the ratio.

	Ratio of the rise to the run
Write the ratio as a fraction.	$\frac{\text{rise}}{\text{run}}$
Substitute in the given values.	$\frac{1\text{ inch}}{1\text{ foot}}$
Convert 1 foot to inches.	$\frac{1\text{ inch}}{12\text{ inches}}$
Simplify, dividing out common factors and units.	$\frac{1}{12}$

So the ratio of rise to run is 1 to 12. This means that the ramp should rise 1 inch for every 12 inches of horizontal run to comply with the guidelines.

Find Unit Rates [footnote]

Section material derived from Openstax Prealgebra: Decimals-Ratios and Rate

In the last example, we calculated that Bob was driving at a rate of $\frac{175 \text{ miles}}{3 \text{ hours}}$. This tells us that every three hours, Bob will travel 175 miles. This is correct, but not very useful. We usually want the rate to reflect the number of miles in one hour. A rate that has a denominator of 1 unit is referred to as a **unit rate**.

Note:
A **unit rate** is a rate with denominator of 1 unit.

Unit rates are very common in our lives. For example, when we say that we are driving at a speed of 68 miles per hour we mean that we travel 68 miles in 1 hour. We would write this rate as 68 miles/hour (read 68 miles per hour). The common abbreviation for this is 68 mph. Note that when no number is written before a unit, it is assumed to be 1.

So 68 miles/hour really means $68 \text{ miles}/1 \text{ hour}$.

Two rates we often use when driving can be written in different forms, as shown:

Example	Rate	Write	Abbreviate	Read
68 miles in 1 hour	$\frac{68 \text{ miles}}{1 \text{ hour}}$	68 miles/hour	68 mph	68 miles per hour
36 miles to 1 gallon	$\frac{36 \text{ miles}}{1 \text{ gallon}}$	36 miles/gallon	36 mpg	36 miles per gallon

Another example of unit rate that you may already know about is hourly pay rate. It is usually expressed as the amount of money earned for one hour of work. For example, if you are paid $12.50 for each hour you work, you could write that your hourly (unit) pay rate is $\$12.50/\text{hour}$ (read $12.50 per hour.)

To convert a rate to a unit rate, we divide the numerator by the denominator. This gives us a denominator of 1.

Example:
Determining a Rate
Exercise:

Problem:

Anita was paid $\$384$ last week for working 32 hours. What is Anita's hourly pay rate?

Solution:
Solution

Start with a rate of dollars to hours. Then divide.	$\$384$ last week for 32 hours
Write as a rate.	$\frac{\$384}{32 \text{ hours}}$
Divide the numerator by the denominator.	$\frac{\$12}{1 \text{ hour}}$
Rewrite as a rate.	$\$12/\text{hour}$

Anita's hourly pay rate is $\$12$ per hour.

Example:
Determining a Rate
Exercise:

Problem:

Sven drives his car 455 miles, using 14 gallons of gasoline. How many miles per gallon does his car get?

Solution:
Solution

Start with a rate of miles to gallons. Then divide.

	455 miles to 14 gallons of gas
Write as a rate.	$\frac{455 \text{ miles}}{14 \text{ gallons}}$
Divide 455 by 14 to get the unit rate.	$\frac{32.5 \text{ miles}}{1 \text{ gallon}}$

Sven's car gets 32.5 miles/gallon, or 32.5 mpg.

Make Unit Conversions in the U.S. System [footnote]

Section material derived from Openstax Prealgebra: The Properties of Real Numbers-Systems of Measurement

There are two systems of measurement commonly used around the world. Most countries use the metric system. The United States uses a different system of measurement, usually called the U.S. system. We will look at the U.S. system first.

The U.S. system of measurement uses units of inch, foot, yard, and mile to measure length and pound and ton to measure weight. For capacity, the units used are cup, pint, quart and gallons. Both the U.S. system and the metric system measure time in seconds, minutes, or hours.

The equivalencies among the basic units of the U.S. system of measurement are listed in [link]. The table also shows, in parentheses, the common abbreviations for each measurement.

U.S. System Units	
Length	**Volume**
1 foot (ft) = 12 inches (in) 1 yard (yd) = 3 feet (ft) 1 mile (mi) = 5280 feet (ft)	3 teaspoons (t) = 1 tablespoon (T) 16 Tablespoons (T) = 1 cup (C) 1 cup (C) = 8 fluid ounces (fl oz) 1 pint (pt) = 2 cups (C) 1 quart (qt) = 2 pints (pt) 1 gallon (gal) = 4 quarts (qt)
Weight	**Time**
1 pound (lb) = 16 ounces (oz) 1 ton = 2000 pounds (lb)	1 minute (min) = 60 seconds (s) 1 hour (h) = 60 minutes (min) 1 day = 24 hours (h) 1 week (wk) = 7 days 1 year (yr) = 365 days

In many real-life applications, we need to convert between units of measurement. We will use the identity property of multiplication to do these conversions. We'll restate the Identity Property of Multiplication here for easy reference.

Equation:

$$\text{For any real number } a, \qquad a \cdot 1 = a \qquad 1 \cdot a = a$$

To use the identity property of multiplication, we write 1 in a form that will help us convert the units. For example, suppose we want to convert inches to feet. We know that 1 foot is equal to 12 inches, so we can write 1 as the fraction $\frac{1 \text{ ft}}{12 \text{ in}}$. When we multiply by this fraction, we do not change the value but just change the units.

But $\frac{12 \text{ in}}{1 \text{ ft}}$ also equals 1. How do we decide whether to multiply by $\frac{1 \text{ ft}}{12 \text{ in}}$ or $\frac{12 \text{ in}}{1 \text{ ft}}$? We choose the fraction that will make the units we want to convert *from* divide out. For example, suppose we wanted to convert 60 inches to feet. If we choose the fraction that has inches in the denominator, we can eliminate the inches.

Equation:

$$60 \cancel{\text{ in}} \cdot \frac{1 \text{ ft}}{12 \cancel{\text{ in}}} = 5 \text{ ft}$$

On the other hand, if we wanted to convert 5 feet to inches, we would choose the fraction that has feet in the denominator.

Equation:

$$5\text{ ft}\cdot\frac{12\text{ in}}{1\cancel{\text{ ft}}}=60\text{ in}$$

We treat the unit words like factors and ‘divide out’ common units like we do common factors.

Note:

Multiply the measurement to be converted by 1; write 1 as a fraction relating the units given and the units needed.
Multiply.
Simplify the fraction, performing the indicated operations and removing the common units.

Example:
Unit Conversions
Exercise:

Problem: Mary Anne is 66 inches tall. What is her height in feet?

Solution:
Solution

Convert 66 inches into feet.	
Multiply the measurement to be converted by 1.	$66\text{ inches}\cdot 1$
Write 1 as a fraction relating the units given and the units needed.	$66\text{ inches}\cdot\frac{1\text{ foot}}{12\text{ inches}}$

Multiply.	$\frac{66 \text{ inches} \cdot 1 \text{ foot}}{12 \text{ inches}}$
Simplify the fraction.	$\frac{66 \cancel{\text{inches}} \cdot 1 \text{ foot}}{12 \cancel{\text{inches}}}$
	$\frac{66 \text{ feet}}{12}$
	5.5 feet

Notice that the when we simplified the fraction, we first divided out the inches.

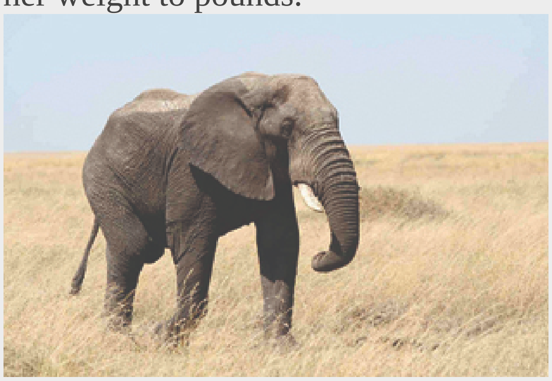
Mary Anne is 5.5 feet tall.

When we use the Identity Property of Multiplication to convert units, we need to make sure the units we want to change from will divide out. Usually this means we want the conversion fraction to have those units in the denominator.

Example:
Unit Conversions
Exercise:

Problem:

Ndula, an elephant at the San Diego Safari Park, weighs almost 3.2 tons. Convert her weight to pounds.



(credit: Guldo Da Rozze, Flickr)

Solution:
Solution

We will convert 3.2 tons into pounds, using the equivalencies in [link]. We will use the Identity Property of Multiplication, writing 1 as the fraction $\frac{2000\text{ pounds}}{1\text{ ton}}$.

	3.2 tons
Multiply the measurement to be converted by 1.	$3.2\text{ tons}\cdot 1$
Write 1 as a fraction relating tons and pounds.	$3.2\text{ tons}\cdot\frac{2000\text{ lbs}}{1\text{ ton}}$
Simplify.	$\frac{3.2\cancel{\text{ tons}}\cdot 2000\text{ lbs}}{1\cancel{\text{ ton}}}$
Multiply.	6400 lbs
	Ndula weighs almost 6,400 pounds.

Sometimes to convert from one unit to another, we may need to use several other units in between, so we will need to multiply several fractions.

Example:
Unit Conversions
Exercise:

Problem:

Juliet is going with her family to their summer home. She will be away for 9 weeks. Convert the time to minutes.

Solution:
Solution

To convert weeks into minutes, we will convert weeks to days, days to hours, and then hours to minutes. To do this, we will multiply by conversion factors of 1.

	9 weeks
Write 1 as $\frac{7\text{ days}}{1\text{ week}}, \frac{24\text{ hours}}{1\text{ day}}, \frac{60\text{ minutes}}{1\text{ hour}}$.	$\frac{9\text{ wk}}{1} \cdot \frac{7\text{ days}}{1\text{ wk}} \cdot \frac{24\text{ hr}}{1\text{ day}} \cdot \frac{60\text{ min}}{1\text{ hr}}$
Cancel common units.	$\frac{9\,\cancel{\text{wk}}}{1} \cdot \frac{7\,\cancel{\text{days}}}{1\,\cancel{\text{wk}}} \cdot \frac{24\,\cancel{\text{hr}}}{1\,\cancel{\text{day}}} \cdot \frac{60\text{ min}}{1\,\cancel{\text{hr}}}$
Multiply.	$\frac{9\cdot7\cdot24\cdot60\text{ min}}{1\cdot1\cdot1\cdot1} = 90,720\text{ min}$
	Juliet will be away for 90,720 minutes.

Example:
Unit Conversions
Exercise:

Problem: How many fluid ounces are in 1 gallon of milk?

(credit: www.bluewaikiki.com, Flickr)

Solution:
Solution

Use conversion factors to get the right units: convert gallons to quarts, quarts to pints, pints to cups, and cups to fluid ounces.

	1 gallon
Multiply the measurement to be converted by 1.	$\frac{1\text{ gal}}{1} \cdot \frac{4\text{ qt}}{1\text{ gal}} \cdot \frac{2\text{ pt}}{1\text{ qt}} \cdot \frac{2\text{ C}}{1\text{ pt}} \cdot \frac{8\text{ fl oz}}{1\text{ C}}$
Simplify.	$\frac{1\ \cancel{\text{gal}}}{1} \cdot \frac{4\ \cancel{\text{qt}}}{1\ \cancel{\text{gal}}} \cdot \frac{2\ \cancel{\text{pt}}}{1\ \cancel{\text{qt}}} \cdot \frac{2\ \cancel{\text{C}}}{1\ \cancel{\text{pt}}} \cdot \frac{8\text{ fl oz}}{1\ \cancel{\text{C}}}$
Multiply.	$\frac{1 \cdot 4 \cdot 2 \cdot 2 \cdot 8\text{ fl oz}}{1 \cdot 1 \cdot 1 \cdot 1 \cdot 1}$
Simplify.	128 fluid ounces
	There are 128 fluid ounces in a gallon.

Make Unit Conversions in the Metric System [footnote]

Section material derived from Openstax Prealgebra: The Properties of Real Numbers-Systems of Measurement

In the metric system, units are related by powers of 10. The root words of their names reflect this relation. For example, the basic unit for measuring length is a meter. One kilometer is 1000 meters; the prefix *kilo*- means thousand. One centimeter is $\frac{1}{100}$ of a meter, because the prefix *centi*- means one one-hundredth (just like one cent is $\frac{1}{100}$ of one dollar).

The equivalencies of measurements in the metric system are shown in [link]. The common abbreviations for each measurement are given in parentheses.

Metric Measurements		
Length	**Mass**	**Volume/Capacity**
1 kilometer (km) = 1000 m 1 hectometer (hm) = 100 m 1 dekameter (dam) = 10 m 1 meter (m) = 1 m 1 decimeter (dm) = 0.1 m 1 centimeter (cm) = 0.01 m 1 millimeter (mm) = 0.001 m	1 kilogram (kg) = 1000 g 1 hectogram (hg) = 100 g 1 dekagram (dag) = 10 g 1 gram (g) = 1 g 1 decigram (dg) = 0.1 g 1 centigram (cg) = 0.01 g 1 milligram (mg) = 0.001 g	1 kiloliter (kL) = 1000 L 1 hectoliter (hL) = 100 L 1 dekaliter (daL) = 10 L 1 liter (L) = 1 L 1 deciliter (dL) = 0.1 L 1 centiliter (cL) = 0.01 L 1 milliliter (mL) = 0.001 L
1 meter = 100 centimeters 1 meter = 1000 millimeters	1 gram = 100 centigrams 1 gram = 1000 milligrams	1 liter = 100 centiliters 1 liter = 1000 milliliters

To make conversions in the metric system, we will use the same technique we did in the U.S. system. Using the identity property of multiplication, we will multiply by a conversion factor of one to get to the correct units.

Have you ever run a 5 k or 10 k race? The lengths of those races are measured in kilometers. The metric system is commonly used in the United States when talking about the length of a race.

Example:
Unit Conversions
Exercise:

Problem: Nick ran a 10-kilometer race. How many meters did he run?

(credit: William Warby, Flickr)

Solution:
Solution

We will convert kilometers to meters using the Identity Property of Multiplication and the equivalencies in [link].

	10 kilometers
Multiply the measurement to be converted by 1.	$10\text{ km} \cdot 1$

Write 1 as a fraction relating kilometers and meters.	$10\text{ km} \cdot \frac{1000\text{ m}}{1\text{ km}}$
Simplify.	$\frac{10\,\cancel{\text{km}} \cdot 1000\text{ m}}{1\,\cancel{\text{km}}}$
Multiply.	10,000 m
	Nick ran 10,000 meters.

Example:
Unit Conversions
Exercise:

Problem:

Eleanor's newborn baby weighed 3200 grams. How many kilograms did the baby weigh?

Solution:
Solution

We will convert grams to kilograms.

	3200 grams
Multiply the measurement to be converted by 1.	$3200\text{ g} \cdot 1$

Write 1 as a fraction relating kilograms and grams.	$3200\text{ g}\cdot\frac{1\text{ kg}}{1000\text{ g}}$
Simplify.	$3200\,\cancel{\text{g}}\cdot\frac{1\text{ kg}}{1000\,\cancel{\text{g}}}$
Multiply.	$\frac{3200\text{ kilograms}}{1000}$
Divide.	3.2 kilograms
	The baby weighed 3.2 kilograms.

Since the metric system is based on multiples of ten, conversions involve multiplying by multiples of ten. In Decimal Operations, we learned how to simplify these calculations by just moving the decimal.

To multiply by $10, 100$, or 1000, we move the decimal to the right $1, 2$, or 3 places, respectively. To multiply by $0.1, 0.01$, or 0.001 we move the decimal to the left $1, 2$, or 3 places respectively.

We can apply this pattern when we make measurement conversions in the metric system.

In [link], we changed 3200 grams to kilograms by multiplying by $\frac{1}{1000}$ $(\text{or } 0.001)$. This is the same as moving the decimal 3 places to the left.

$$3200\cdot\frac{1}{1000} \qquad 3200.$$

$$3.2 \qquad 3.2$$

Example:
Unit Conversions
Exercise:

Problem: Convert: ⓐ 350 liters to kiloliters ⓑ 4.1 liters to milliliters.

Solution:
Solution

ⓐ We will convert liters to kiloliters. In [link], we see that $1\text{ kiloliter} = 1000\text{ liters}.$

	350 L
Multiply by 1, writing 1 as a fraction relating liters to kiloliters.	$350\text{ L}\cdot\frac{1\text{ kL}}{1000\text{ L}}$
Simplify.	$350\,\cancel{\text{L}}\cdot\frac{1\text{ kL}}{1000\,\cancel{\text{L}}}$
Move the decimal 3 units to the left.	$350\,\cancel{\text{L}}\cdot\frac{1\text{ kL}}{1000\,\cancel{\text{L}}}$
	0.35 kL

ⓑ We will convert liters to milliliters. In [link], we see that $1\text{ liter} = 1000\text{ milliliters}.$

	4.1 L
Multiply by 1, writing 1 as a fraction relating milliliters to liters.	$4.1\text{ L}\cdot\dfrac{1000\text{ mL}}{1\text{ L}}$
Simplify.	$4.1\cancel{\text{L}}\cdot\dfrac{1000\text{ mL}}{1\cancel{\text{L}}}$
Move the decimal 3 units to the left.	4.100 mL
	4100 mL

Convert Between U.S. and Metric Systems of Measurement [footnote]

Section material derived from Openstax Prealgebra: The Properties of Real Numbers-Systems of Measurement

Many measurements in the United States are made in metric units. A drink may come in 2-liter bottles, calcium may come in 500-mg capsules, and we may run a 5-K race. To work easily in both systems, we need to be able to convert between the two systems.

[link] shows some of the most common conversions.

Conversion Factors Between U.S. and Metric Systems		
Length	**Weight**	**Volume**

Conversion Factors Between U.S. and Metric Systems		
Length	**Weight**	**Volume**
1 in = 2.54 cm 1 ft = 0.305 m 1 yd = 0.914 m 1 mi = 1.61 km 1 m = 3.28 ft	1 lb = 0.45 kg 1 oz = 28 g 1 kg = 2.2 lb	1 qt = 0.95 L 1 fl oz = 30 mL 1 L = 1.06 qt

We make conversions between the systems just as we do within the systems—by multiplying by unit conversion factors.

Example:
Unit Conversions
Exercise:

Problem:

Lee's water bottle holds 500 mL of water. How many fluid ounces are in the bottle? Round to the nearest tenth of an ounce.

Solution:
Solution

	500 mL
Multiply by a unit conversion factor relating mL and ounces.	$500\text{ mL}\cdot\frac{1\text{ fl oz}}{30\text{ mL}}$
Simplify.	$\frac{500\text{ fl oz}}{30}$
Divide.	16.7 fl. oz.

	The water bottle holds 16.7 fluid ounces.

The conversion factors in [link] are not exact, but the approximations they give are close enough for everyday purposes. In [link], we rounded the number of fluid ounces to the nearest tenth.

Example:
Unit Conversions
Exercise:

Problem:

Soleil lives in Minnesota but often travels in Canada for work. While driving on a Canadian highway, she passes a sign that says the next rest stop is in 100 kilometers. How many miles until the next rest stop? Round your answer to the nearest mile.

Solution:
Solution

	100 kilometers
Multiply by a unit conversion factor relating kilometers and miles.	$100 \text{ kilometers} \cdot \frac{1 \text{ mile}}{1.61 \text{ kilometers}}$ $100 \cdot \frac{1 \text{ mi}}{1.61 \text{ km}}$
Simplify.	$\frac{100 \text{ mi}}{1.61}$
Divide.	62 mi
	It is about 62 miles to the next rest stop.

Key Concepts

- **Making Unit Conversions**
  - Multiply the measurement to be converted by 1; write 1 as a fraction relating the units given and the units needed.
  - Multiply.
  - Simplify the fraction, performing the indicated operations and removing the common units.
- **Conversions within US units**

U.S. System Units	
Length	**Volume**
1 foot (ft) = 12 inches (in) 1 yard (yd) = 3 feet (ft) 1 mile (mi) = 5280 feet (ft)	3 teaspoons (t) = 1 tablespoon (T) 16 Tablespoons (T) = 1 cup (C) 1 cup (C) = 8 fluid ounces (fl oz) 1 pint (pt) = 2 cups (C) 1 quart (qt) = 2 pints (pt) 1 gallon (gal) = 4 quarts (qt)
Weight	**Time**
1 pound (lb) = 16 ounces (oz) 1 ton = 2000 pounds (lb)	1 minute (min) = 60 seconds (s) 1 hour (h) = 60 minutes (min) 1 day = 24 hours (h) 1 week (wk) = 7 days 1 year (yr) = 365 days

- **Converting within the metric system**

Metric Measurements		
Length	**Mass**	**Volume/Capacity**
1 kilometer (km) = 1000 m 1 hectometer (hm) = 100 m 1 dekameter (dam) = 10 m 1 meter (m) = 1 m 1 decimeter (dm) = 0.1 m 1 centimeter (cm) = 0.01 m 1 millimeter (mm) = 0.001 m	1 kilogram (kg) = 1000 g 1 hectogram (hg) = 100 g 1 dekagram (dag) = 10 g 1 gram (g) = 1 g 1 decigram (dg) = 0.1 g 1 centigram (cg) = 0.01 g 1 milligram (mg) = 0.001 g	1 kiloliter (kL) = 1000 L 1 hectoliter (hL) = 100 L 1 dekaliter (daL) = 10 L 1 liter (L) = 1 L 1 deciliter (dL) = 0.1 L 1 centiliter (cL) = 0.01 L 1 milliliter (mL) = 0.001 L
1 meter = 100 centimeters 1 meter = 1000 millimeters	1 gram = 100 centigrams 1 gram = 1000 milligrams	1 liter = 100 centiliters 1 liter = 1000 milliliters

Glossary

Metric System
: a system of measurement in which units are related by powers of 10.

U.S. System
: a system of measurement used in the United States.

Ratio
: A ratio compares two numbers or two quantities that are measured with the same unit. The ratio of a to b is written a to b, $\frac{a}{b}$, or $a : b$.

Rate
: A rate compares two quantities of different units. A rate is usually written as a fraction.

Unit Rate
: A unit rate is a rate with denominator of 1 unit.

Weighted Average

An average whose data values are multipled by a predetermined weight before calculation of teh average.

Weighted Averages - Lessons 5.B - 6.B
This module correlates to our Corequisite for MAT 1043(QR) and NCBO.

Note:
In case you missed something in class, or just want to review a specific topic covered in this Module, here is a list of topics covered:

Weighted Averages [footnote]

This material was created by Amanda Towry using Microsoft Word processing program.

A weighted average is an average that has weights attached to the data values based on how important or relevant they are before taking the average. Not all data should be equally weighted when calculating the average. For example, When looking at poll numbers for politicians over various companies, a weighted average may be taken because sample size and date they were administered must be taken into account. A poll conducted two years ago is nowhere near as valid as one conducted last week, and a sample size of 100 is much less reliable than a size of 10,000. Your grade in your classes is a weighted average.

In a weighted average, some values contribute to the average more than others. This is often determined by how recent or significant the data value is. Any data value given a larger weight will affect the mean more intensely. Thus, when certain values are more weighted than others, the mean of the data can change.

The weights to use are usually provided to you, but in this class, if you are not given the weights to use, you can use the traditional numbering of 1 through the maximum number included in a group of data values.

It is important to remember that if decimal weights are being used, they must all add up to one in total. For example, three data values may be given the weights of $0.25, 0.40,$ and 0.35, because they all add up to 1.0.

Weighted moving averages often use the weighted averages formula, where you take the sum of the data values multiplied by their weights, divided by the sum of the weights.

$$\frac{\sum_{i=1}^{n}(x_i \cdot w_i)}{\sum_{i=1}^{n} w_i}$$

Where $x =$ each data value, $w =$ each weight value, and i is representing each instance in the data sequence, starting at 1. Σ represnts the sum; meaning you add up all of the products of weights and data values.

Example:
Weighted Average
Calculate the weighted mean of the grade a student would receive in their Mathematics course based on the following weights and item grades.

Category	Weight	Score
Preview/Practice assignments average (MML)	20%	92
Student Portfolio	30%	85
Mini-Project average (Blackboard)	20%	89
Exams	30%	78
Average:		???

Mid-Term grade report

Each score should be multiplied by the weight associated with that category. First, each percent should be converted to decimal form.

Category	Weight	Score
Preview/Practice assignments average (MML)	0.20	92
Student Portfolio	0.30	85
Mini-Project average (Blackboard)	0.20	89
Exams	0.30	78
Average:		???

Midterm grade weights converted from percents to decimals.

Notice that all of the category weights add up to 100%, or 1.0. The category weight must be multiplied by the category score to get the weighted score.

Category	Weighted Score
Preview/Practice assignments average (MML)	0.20*92
Student Portfolio	0.30*85
Mini-Project average (Blackboard)	0.20*89
Exams	0.30*78
Average:	

Midterm grades being multiplied by the category weights.

Category	Weighted Score
Preview/Practice assignments average (MML)	18.4
Student Portfolio	25.5
Average:	

Category	Weighted Score
Mini-Project average (Blackboard)	17.8
Exams	23.4
Average:	

Midterm weighted scores after multiplying the category score by the category weights.

To find the final midterm grade, you add together all the weights scores previously calculated and divide by the sum of the weights.
To find the sum of the weights to divide by, we would add all the weights together. As we discussed earlier, they all add up to 1:
Equation:

$$0.20, 0.30, 0.20, 0.30 = 1.0$$

This means that whatever the sum of the weighted scores is, that is the final weighted average. This is because any number divided by 1 is itself.
Equation:

$$\frac{(18.4 + 25.5 + 17.8 + 23.4)}{1.0} = \frac{85.1}{1.0} = 85.1$$

Category	Weighted Score
Preview/Practice assignments average (MML)	18.4
Average: **85.1**	

Category	Weighted Score
Student Portfolio	25.5
Mini-Project average (Blackboard)	17.8
Exams	23.4
Average: **85.1**	

Final calcualted midterm grade using weighted averages.

When all of the weights scores are added together, you arrive at the overall weighted average. Each individual score is weighted differently based on the importance in the coursework.

Expected Value [footnote]

Section material derived from Openstax Introductory Statistics: Discrete Random Variables-Mean or Expected Value and Standard Deviation

The **expected value** is often referred to as the **"long-term" average or mean**. This means that over the long term of doing an experiment over and over, you would **expect** this average.

You toss a coin and record the result. What is the probability that the result is heads? If you flip a coin two times, does probability tell you that these flips will result in one heads and one tail? You might toss a fair coin ten times and record nine heads. As you learned in [link], probability does not describe the short-term results of an experiment. It gives information about what can be expected in the long term. To demonstrate this, Karl Pearson once tossed a fair coin 24,000 times! He recorded the results of each toss, obtaining heads 12,012 times. **In his experiment, Pearson illustrated the Law of Large Numbers**.

The Law of Large Numbers states that, as the number of trials in a probability experiment increases, the difference between the theoretical probability of an event and the relative frequency approaches zero **(the theoretical probability and the relative frequency get closer and closer together)**. When evaluating the long-term results of statistical experiments, we often want to know the "average" outcome. This "long-term average" is known as the **mean** or **expected value** of the experiment and is denoted by the Greek letter μ. In other words, after conducting many trials of an experiment, you would expect this average value.

Note:
To find the expected value or long term average, μ, simply multiply each value of the random variable by its probability and add the products.

Example:
A men's soccer team plays soccer zero, one, or two days a week. The probability that they play zero days is 0.2, the probability that they play one day is 0.5, and the probability that they play two days is 0.3. Find the long-term average or expected value, μ, of the number of days per week the men's soccer team plays soccer.
To do the problem, first let the random variable X = the number of days the men's soccer team plays soccer per week. X takes on the values 0, 1, 2. Construct a PDF table adding a column $x*P(x)$. In this column, you will multiply each x value by its probability.

x	$P(x)$	$x*P(x)$
0	0.2	(0)(0.2) = 0

x	$P(x)$	$x*P(x)$
1	0.5	(1)(0.5) = 0.5
2	0.3	(2)(0.3) = 0.6

Expected Value Table This table is called an expected value table. The table helps you calculate the expected value or long-term average.

Add the last column $x*P(x)$ to find the long term average or expected value: (0)(0.2) + (1)(0.5) + (2)(0.3) = 0 + 0.5 + 0.6 = 1.1.
The expected value is 1.1. The men's soccer team would, on the average, expect to play soccer 1.1 days per week. The number 1.1 is the long-term average or expected value if the men's soccer team plays soccer week after week after week. We say $\mu = 1.1$.

Example:
Suppose you play a game of chance in which five numbers are chosen from 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. A computer randomly selects five numbers from zero to nine with replacement. You pay $2 to play and could profit $100,000 if you match all five numbers in order (you get your $2 back plus $100,000). Over the long term, what is your **expected** profit of playing the game?
To do this problem, set up an expected value table for the amount of money you can profit.
Let X = the amount of money you profit. The values of x are not 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. Since you are interested in your profit (or loss), the values of x are 100,000 dollars and −2 dollars.
To win, you must get all five numbers correct, in order. The probability of choosing one correct number is $\frac{1}{10}$ because there are ten numbers. You may choose a number more than once. The probability of choosing all five numbers correctly and in order is
Equation:

$$\left(\frac{1}{10}\right)\left(\frac{1}{10}\right)\left(\frac{1}{10}\right)\left(\frac{1}{10}\right)\left(\frac{1}{10}\right) = (1)(10^{-5}) = 0.00001.$$

Therefore, the probability of winning is 0.00001 and the probability of losing is

Equation:

$$1 - 0.00001 = 0.99999.$$

The expected value table is as follows:

	x	*P*(*x*)	*x***P*(*x*)
Loss	–2	0.99999	(–2)(0.99999) = –1.99998
Profit	100,000	0.00001	(100000)(0.00001) = 1

Add the last column. –1.99998 + 1 = –0.99998

Since –0.99998 is about –1, you would, on average, expect to lose approximately $1 for each game you play. However, each time you play, you either lose $2 or profit $100,000. The $1 is the average or expected LOSS per game after playing this game over and over.

Example:

Suppose you play a game with a biased coin. You play each game by tossing the coin once. $P(\text{heads}) = \frac{2}{3}$ and $P(\text{tails}) = \frac{1}{3}$. If you toss a head, you pay $6. If you toss a tail, you win $10. If you play this game many times, will you come out ahead?

Exercise:

Problem: a. Define a random variable X.

Solution:

a. X = amount of profit

Exercise:

Problem: b. Complete the following expected value table.

	x	____	____
WIN	10	$\frac{1}{3}$	____
LOSE	____	____	$\frac{-12}{3}$

Solution:

b.

	x	$P(x)$	$xP(x)$
WIN	10	$\frac{1}{3}$	$\frac{10}{3}$

	x	$P(x)$	$xP(x)$
LOSE	–6	$\frac{2}{3}$	$\frac{-12}{3}$

Exercise:

Problem: c. What is the expected value, μ? Do you come out ahead?

Solution:

c. Add the last column of the table. The expected value $\mu = \frac{-2}{3}$. You lose, on average, about 67 cents each time you play the game so you do not come out ahead.

Example:
Find the expected value from the expected value table.

x	$P(x)$	$x*P(x)$
2	0.1	
4	0.3	
6	0.4	
8	0.2	

x	*P*(*x*)	*x*P*(*x*)
2	0.1	2(0.1) = 0.2
4	0.3	4(0.3) = 1.2
6	0.4	6(0.4) = 2.4
8	0.2	8(0.2) = 1.6

The expected value is the sum of the products between the probability of an event and the event variable's value:

Equation:

$$0.2 + 1.2 + 2.4 + 1.6 = 5.4$$

Weighted Moving Averages [footnote]

This material was created by Amanda Towry using Microsoft Excel spreadsheet and Microsoft Word processing programs.

When using a simple average, or the mean, of a data set, all data values are being given equal value. This makes sense when all of the individual data values are equally important, such as average temperature on a specific day or the average number of candies in bags produced in a day at a company production plant. However, there are certain situations where older data or data from smaller sample sizes would be considered less relevant. This is where the weighted average came in the previous module. Let's suppose then, that someone wanted to then look at the trends of change over

extended periods of times of one of those instances requiring weighted averages. Maybe a students test scores over a semester? The sales changes for a company over different financial quarters? How would that best be carried out? This is where the weighted moving average comes in handy. It allows for extended periods of data to be "smoothed" out so that overall trends and behavior can be determined and found within the instantaneous changes from data point to data point.

Weighted moving averages are weighted averages that "move" down sets of data. These moving averages allow for "jagged" data to be smoothed or straightened out so that it is easier to see overall trends of data over time.

A weighted moving average will take groups of values out of a data set, moving down the list one data value at a time, and determine a weighted average for a group of values at each successive step through time. Just as in a normal weighted average, each value in the group is assigned a weighting based on the relevance or age of the data.

The number of data points to include in the groups is commonly three; but for this class, you will always be told how many data values to include in a group.

The weights to use are usually provided to you, but in this class, if you are not given the weights to use, you can use the traditional numbering of 1 through the maximum number included in a group of data values.

Weighted Moving Average Equation

Weighted moving averages still use the weighted averages formula, where you take the sum of the data values multiplied by their weights, divided by the sum of the weights.

$$\frac{\sum_{i=1}^{n}(x_i \cdot w_i)}{\sum_{i=1}^{n} w_i}$$

Where x = each data value, w = each weight value, and i is representing each instance in the data sequence, starting at 1. Σ represnts the sum; meaning you add up all of the products of weights and data values.

When filling in the weighted moving average values, you must be sure to start the data in the column number that matches the number of data values in the grouping. For example, if you are taking groups of four from your data set, the first moving average value will start in the weighted average column and in the fourth row. The following example illustrates this.

Example:
Weighted Moving Average

For the following data, calculate the moving average for groups of two data points at a time. Use the traditional weights of 1 and 2 based off most relevant data values.

Week	Sales
1	39
2	44
3	40
4	45
5	38
6	43

First, we must create a new column to fill in the moving averages.

Week	Sales	Wtd Moving Avg
1	39	
2	44	
3	40	
4	45	
5	38	
6	43	

Since we are taking groups of two, the first new data value will be a weighted average of week 1 and week 2 sales and will be placed in the second row of the last column. Because week 2 is more recent than week 1, week 2 gets the weighting of 2 and week 1 gets the weighting of 1.

Equation:

$$(39\ )(\ 1\ )+(\ 44\ )(\ 2\ )= 39 + 88 = 127$$

Now, when we carry out the division, we must divide by the sum of the weights; which is $1 + 2 = 3$.

Equation:

$$\frac{127}{3} = 42.33$$

This value we have calcualted must be placed in the table. Becuase we are taking groups of two, this first calculated data value must be palced in the second row of the weighted average column. This is because we started with the first two data values and there was no data before them in order to calculate a moving average for the first row in the weighted average column.

Week	Sales	Wtd Moving Avg
1	39	
2	44	42.33
3	40	
4	45	
5	38	
6	43	

Now we repeat the steps for the next data point and place the results in the third row of the last column.

Equation:

$$(44)(1)+(40)(2)=44+80=124$$

Equation:

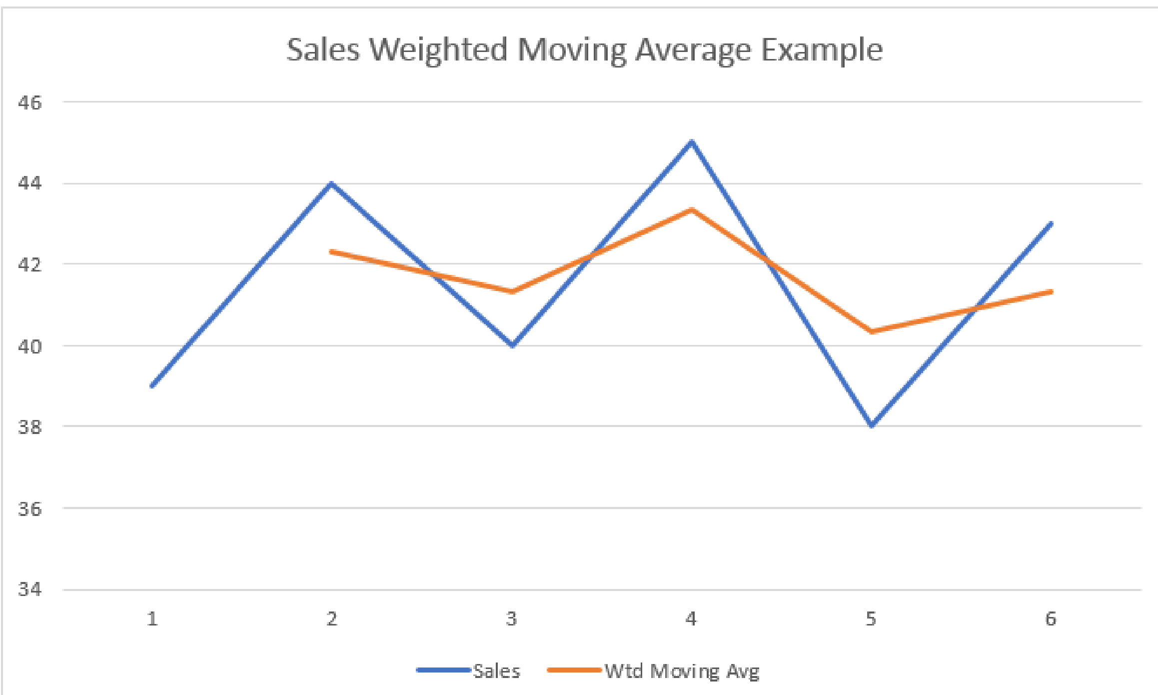
$$\frac{124}{3}=41.33$$

Week	Sales	Wtd Moving Avg
1	39	
2	44	42.333333
3	40	41.333333
4	45	
5	38	
6	43	

Continuing this process results in the following table of values.

Week	Sales	Wtd Moving Avg
1	39	
2	44	42.333333
3	40	41.333333
4	45	43.333333
5	38	40.333333
6	43	41.333333

Look at how the graph of the original sales values compares to the weighted moving average. It is much smoother and easier to notice the trends over time.



Notice that the weighted moving average line (the red line) seems to "lag" behind the actual data (the blue line) in the graph. This is because we started the weighted moving average in the second row and we didn't have a value for the first row in the weighted average data.

Using Spreadsheets for Calculating Statistics [footnote]

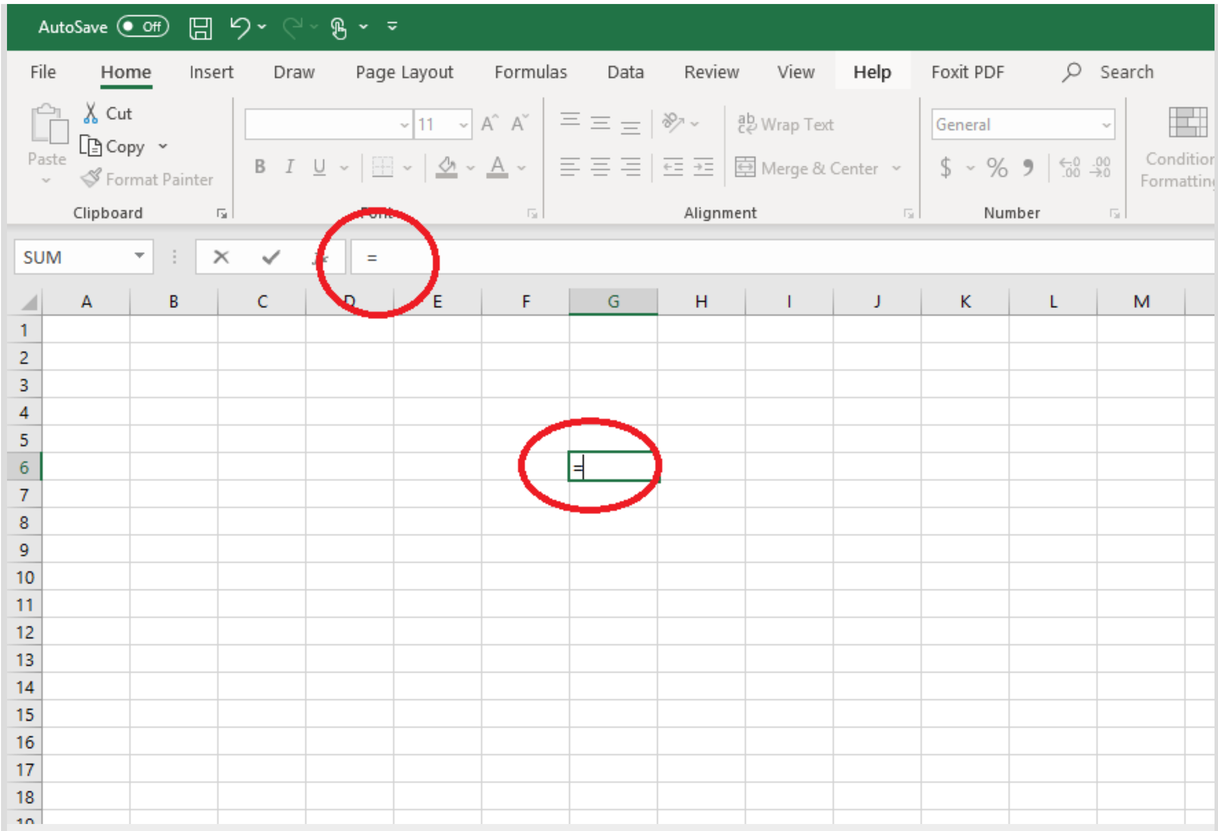
This material was created by Amanda Towry using Microsoft Excel spreadsheets program.

When working with data and statistics, there are often a lot of mathematical calculations that have to be carried out. Depending on how much data needs to be analyzed, it can become tenuous and time-consuming to compute needed statistics by hand. This is why most statistical analysis is done by the use of some kind of spreadsheet software.

In your course, you will be using Microsoft Excel a lot. Many of the calculations you will be expected to calculate can be easily entered into a cell and the program will do the math for you. The most important aspect of your skill, in this case, is knowing how to properly and efficient input data and utilizing the correct equations to complete your data analyses.

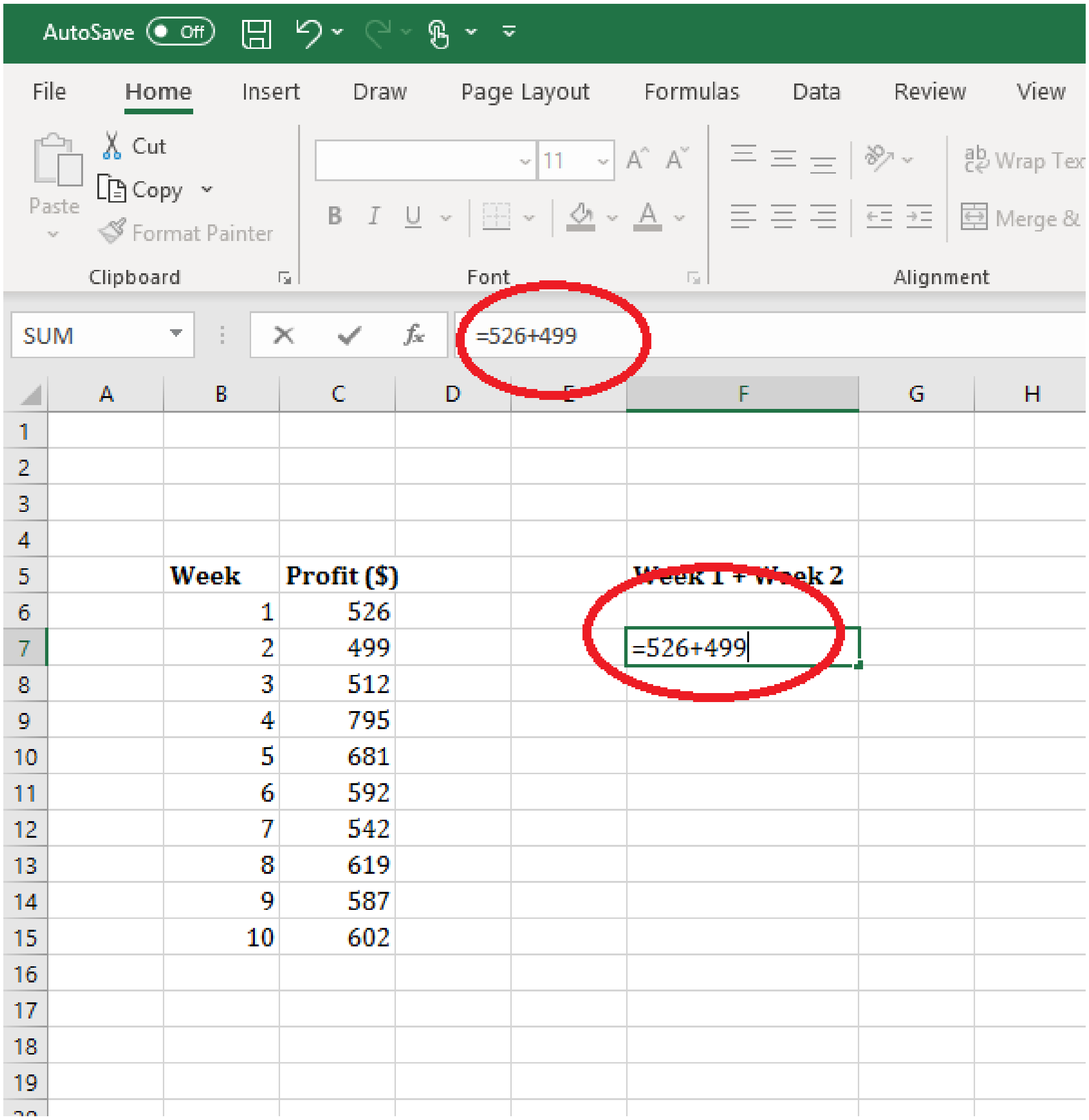
What follows are the many common equations and calculations you will be learning to do in the spreadsheet software you are using in this course.

Note: If you are having the spreadsheet calculate a statistic, it will not know to do so unless the first thing you enter is an equal sign. Anything else after the equal sign the spreadsheet will interpret as an equation or data to use in the equation you cite. When you finish the equation you want to type into the cell, you must hit the enter key for it to calculate the result.

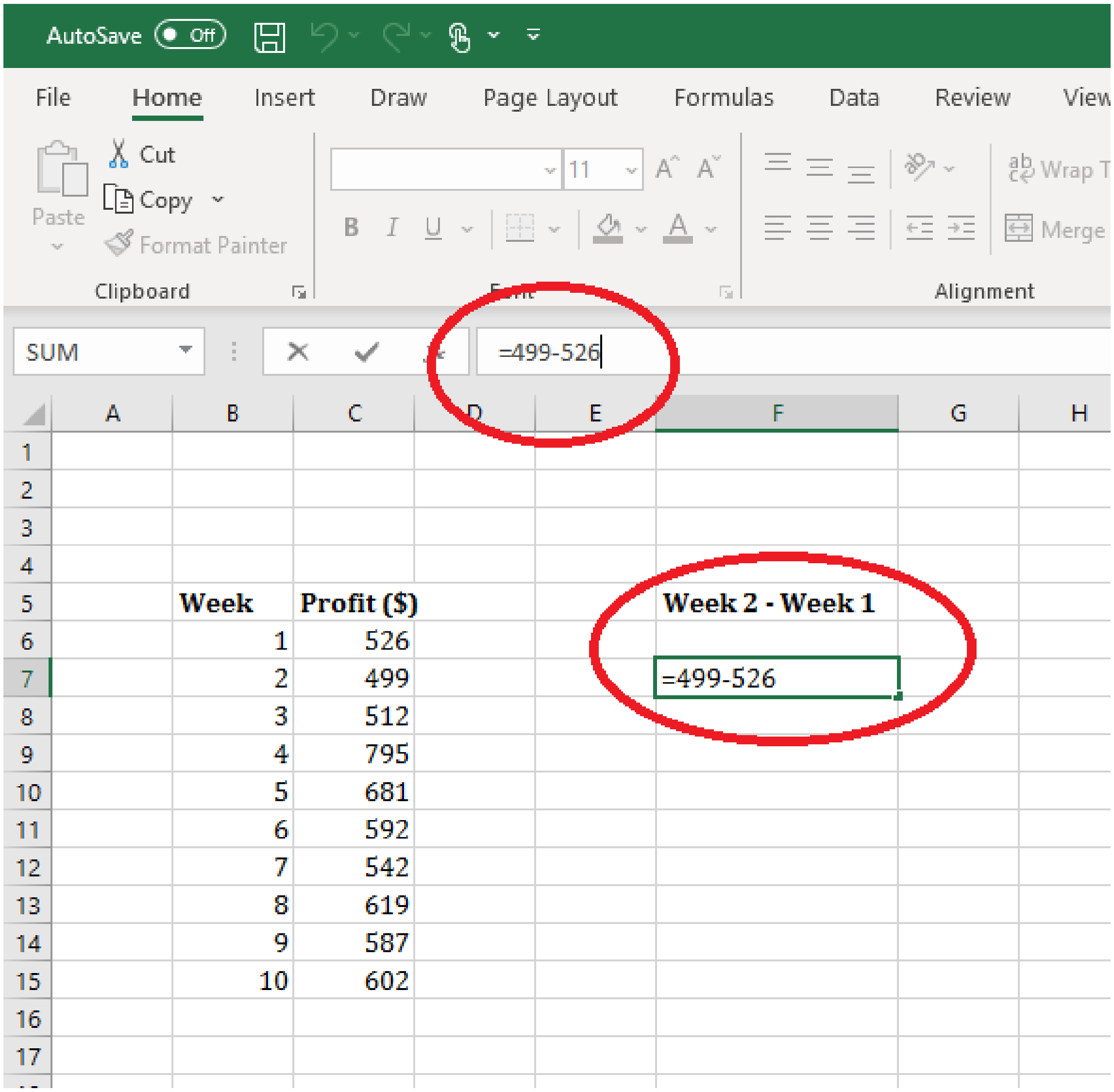


Adding and Subtracting in Spreadsheet
When you need to add or subtract values, there are two options.

- **Manually enter the numbers to be added with addition/subtraction signs included**

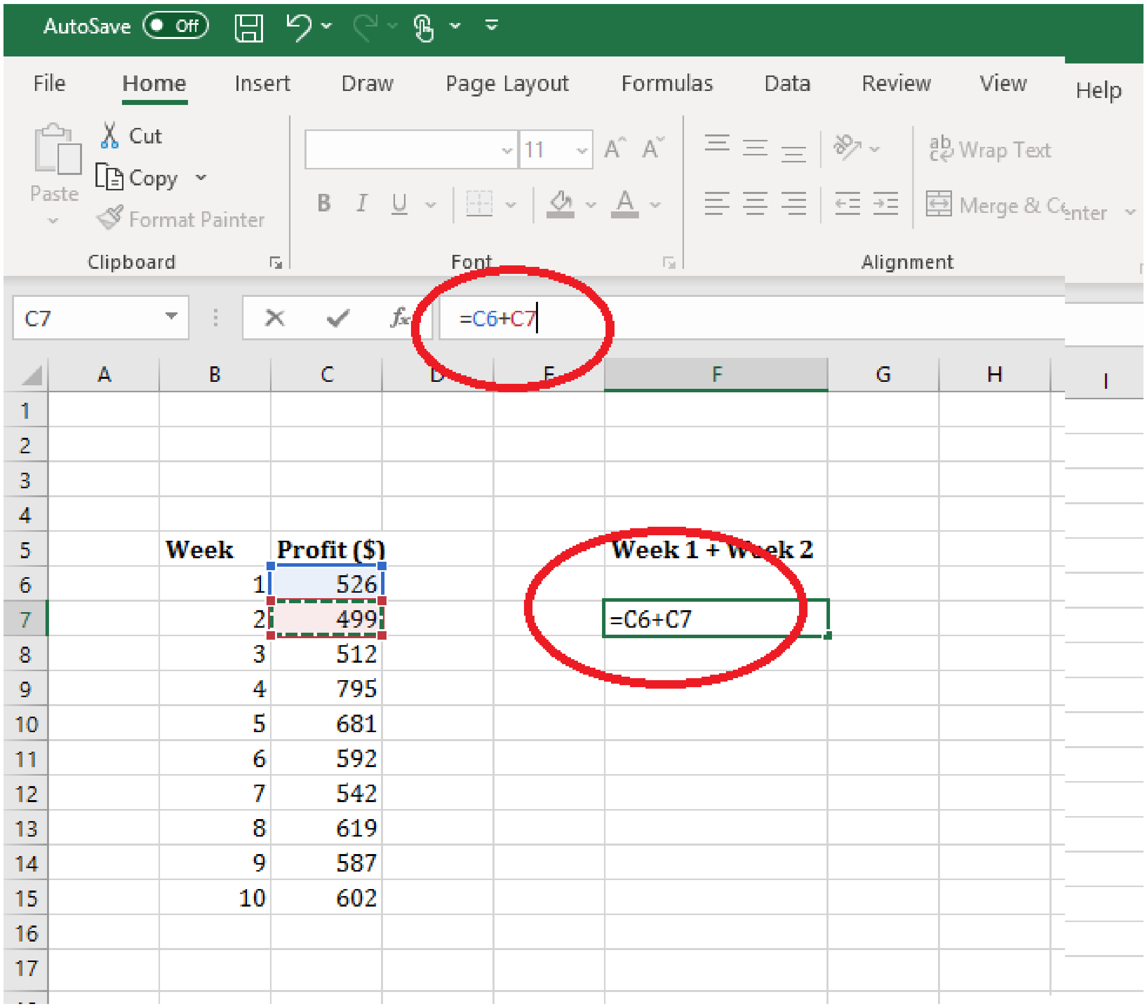


Adding by manually entering values.

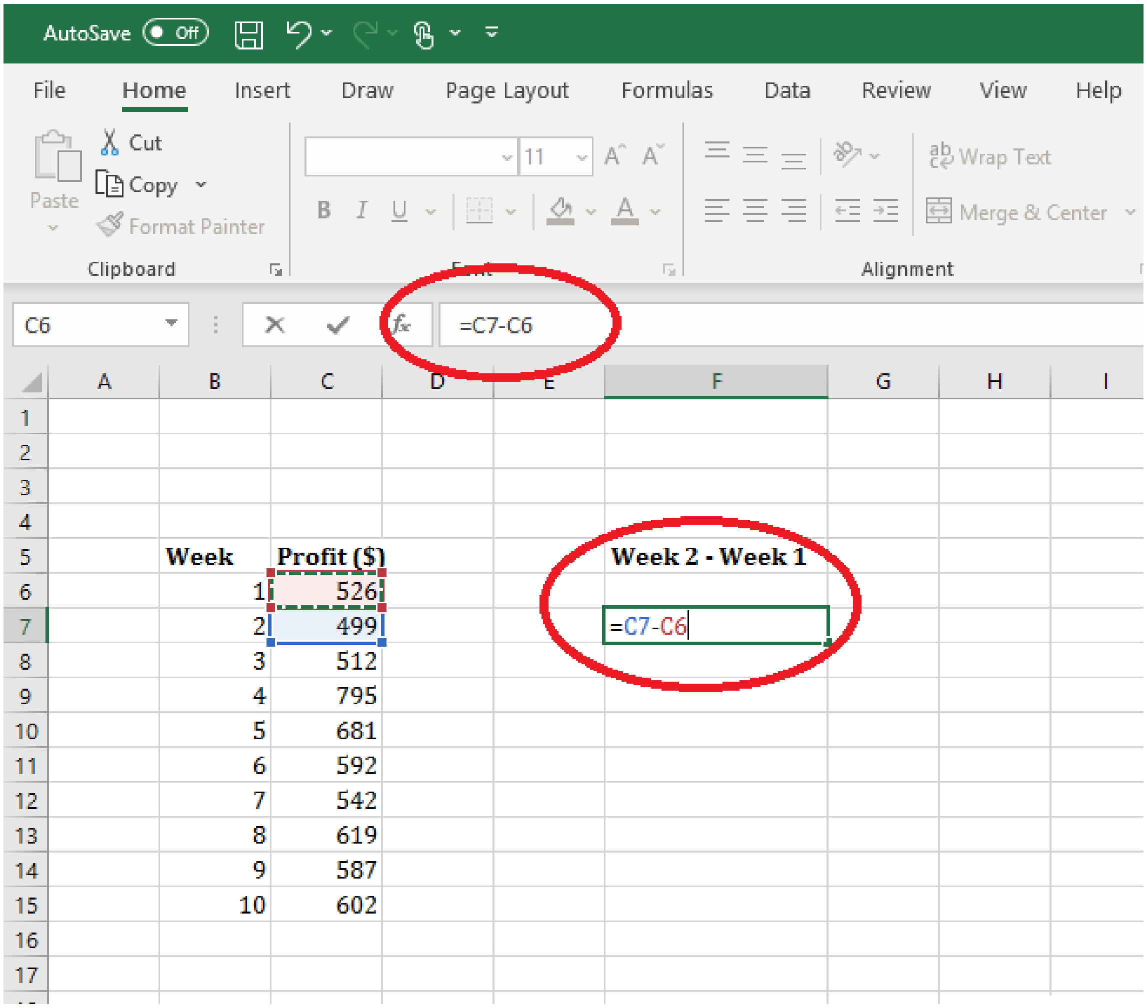


Subtracting by manually entering values.

- **Clicking on cells that include the values that you want to add together with the addition/subtraction sign included**



Adding by selecting cells with the values you want to add within them.Make sure to include the addition sign manually.



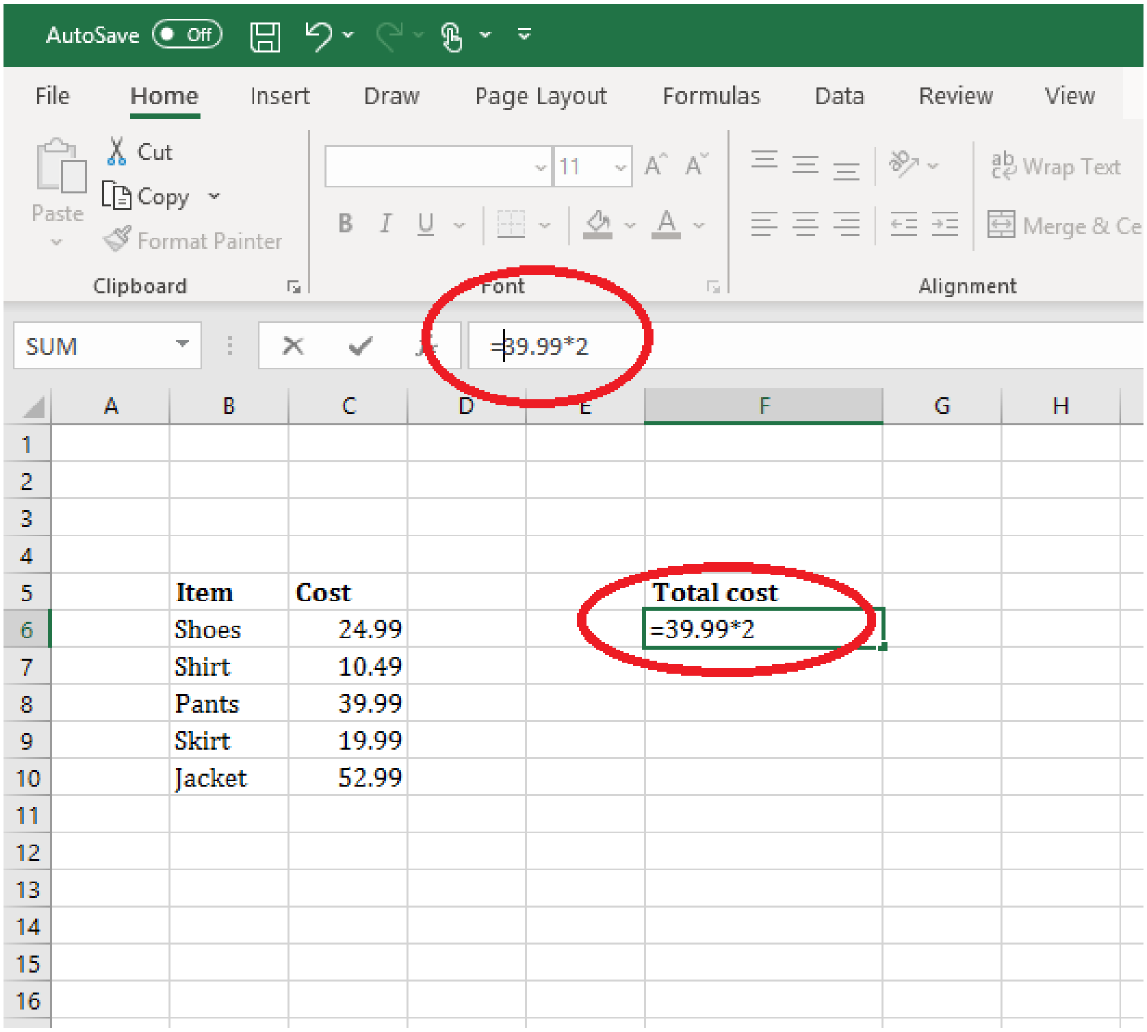
Subtractinging by selecting cells with the values you want to subtract within them. Make sure to include the subtraction sign manually.

Multiplying and Dividing in Spreadsheet

In the spreadsheet programs you use, you may need to multiply or divide two numbers. Spreadsheets do not use regular multiplication or division symbols. Adding an x will not perform the multiplication function in spreadsheets. For multiplication, you must use an asterisk * to complete multiplication between numbers. You can either:

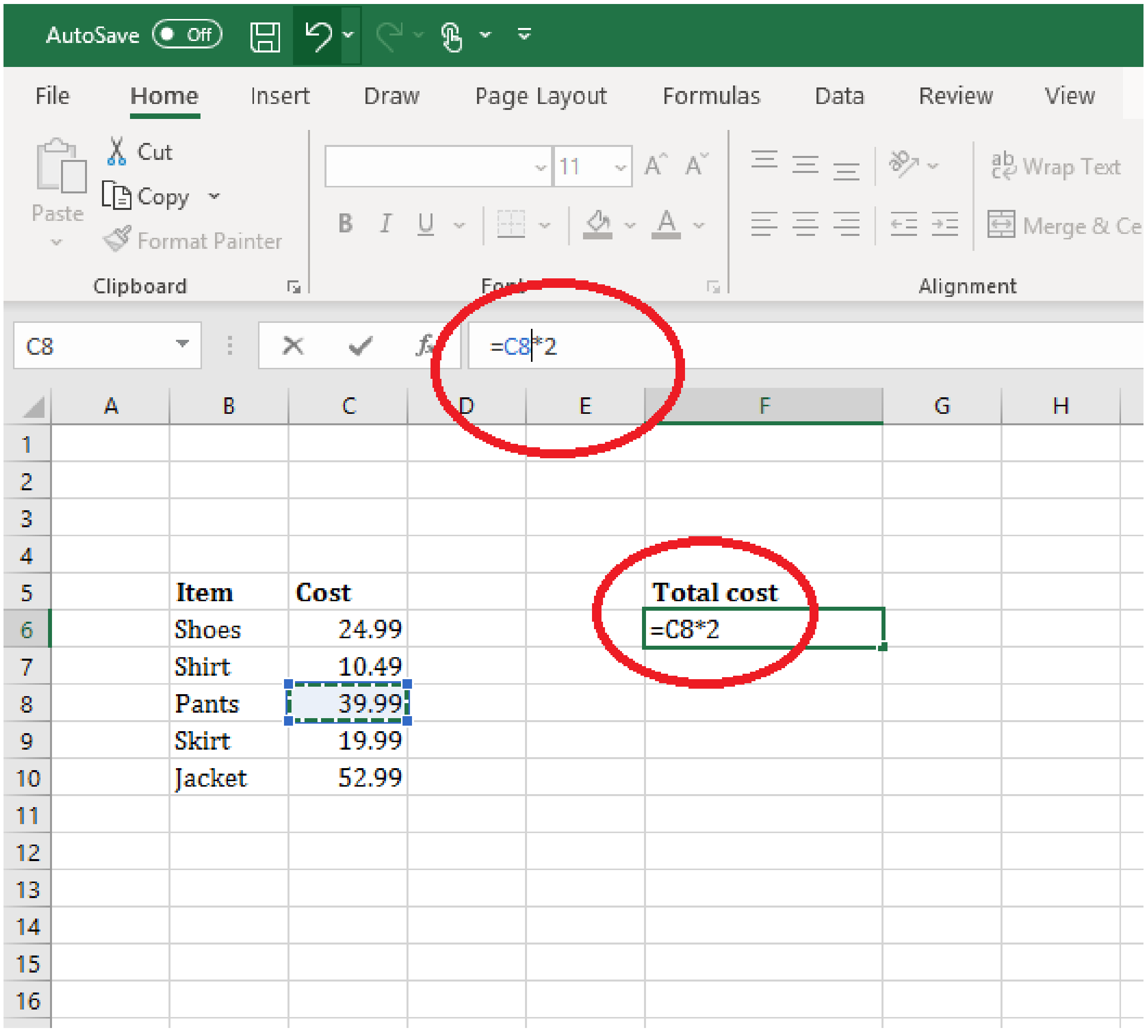
Multiplying

- Manually enter the values to be multiplied with an asterisk between them



Multiplication by manually entering values into a cell with an asterisk for multiplication.

- Click on the cells including the values you want to multiply together with an asterisk between them

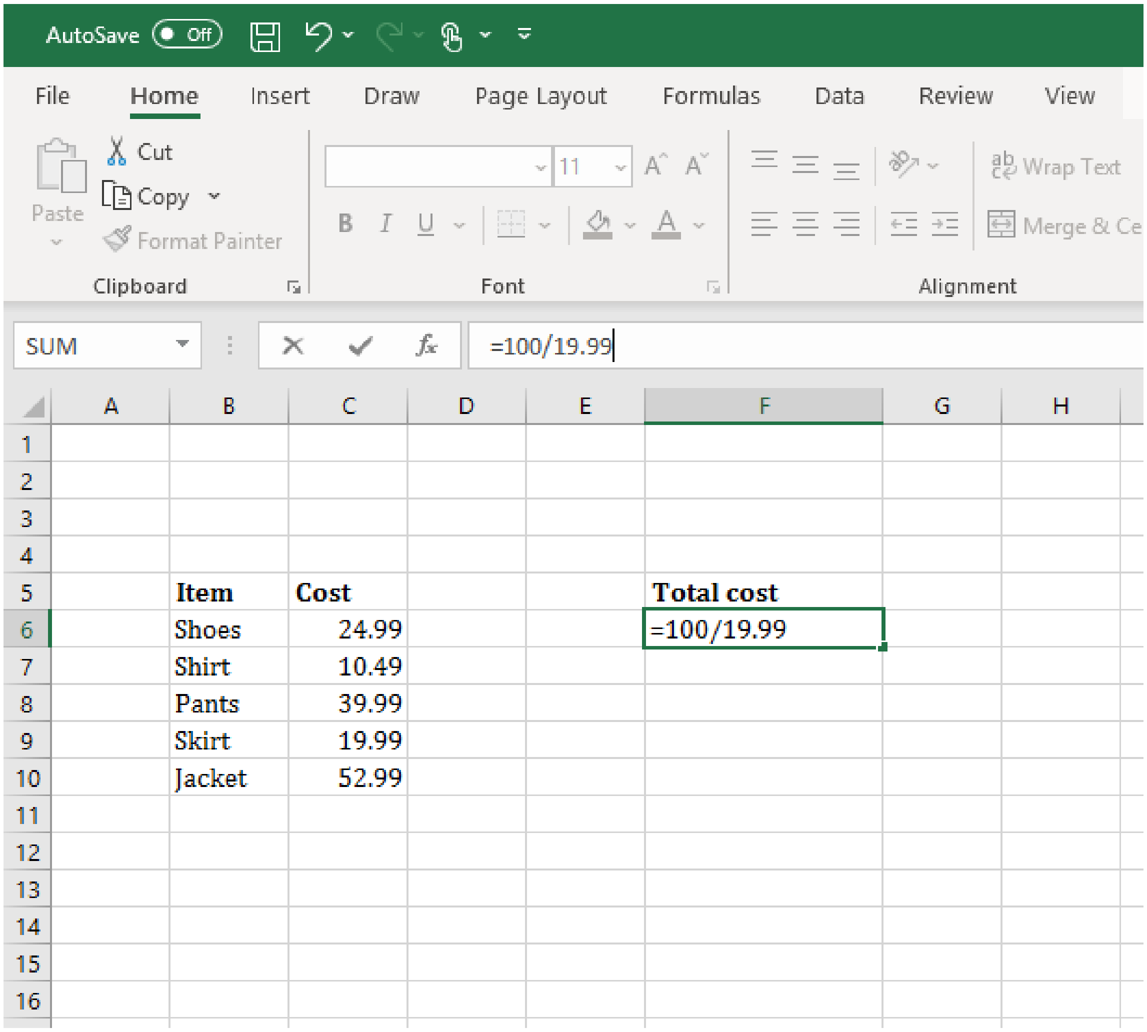


Multiplication by clicking on specific cell values with an asterisk for multiplication.

/ to complete division between two numbers. You can either:

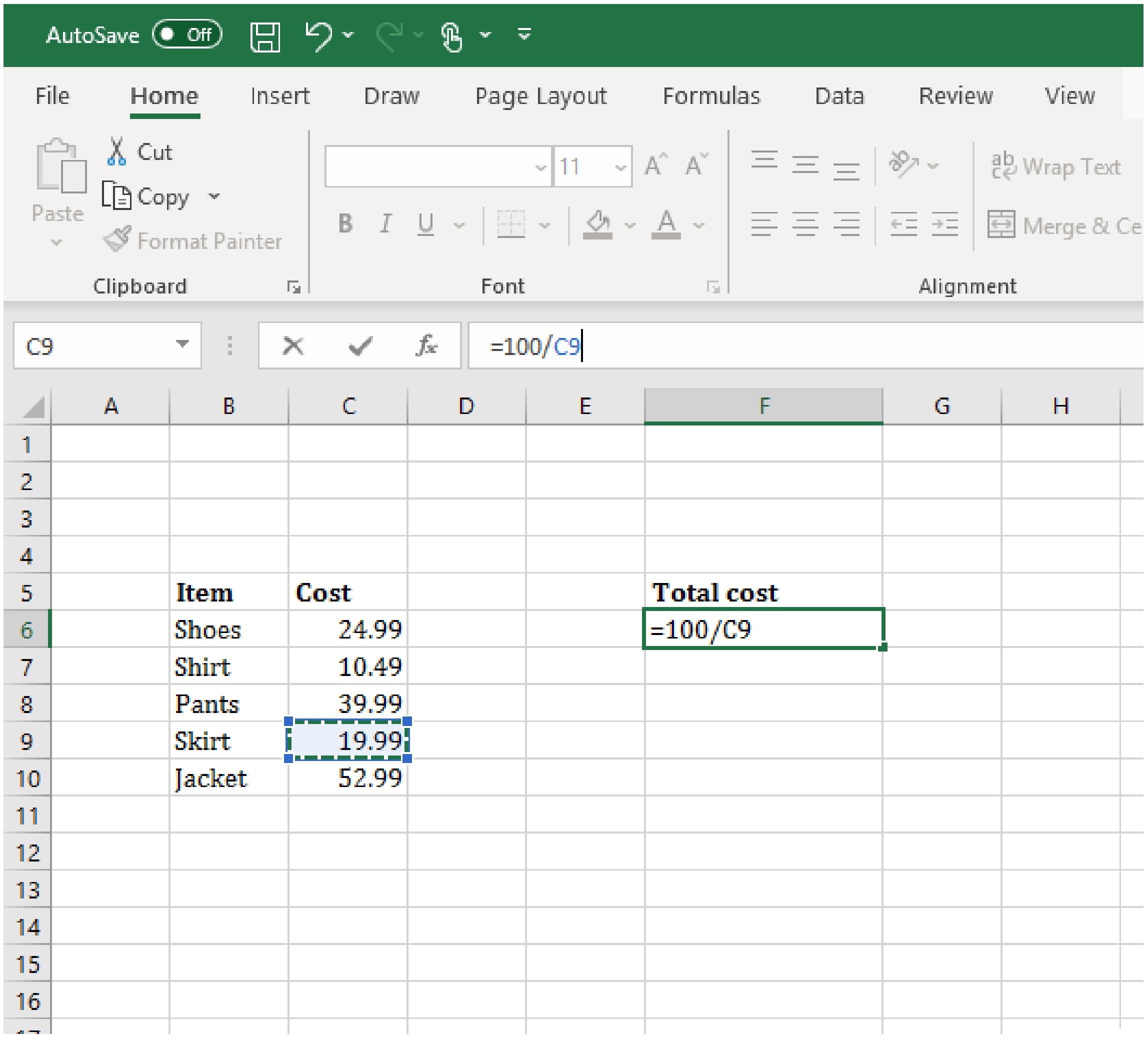
Dividing

- Manually enter the values to be divided with a backslash between them



DIvision by manually entering values into a cell with a backslash for division.

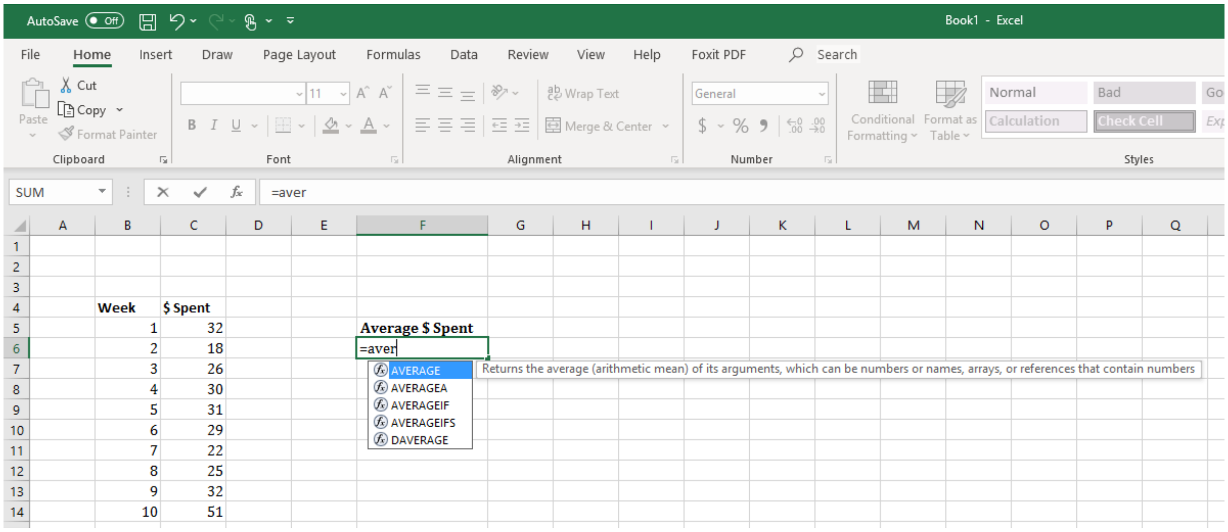
- Click on the cells including the values you want to multiply together with an asterisk between them



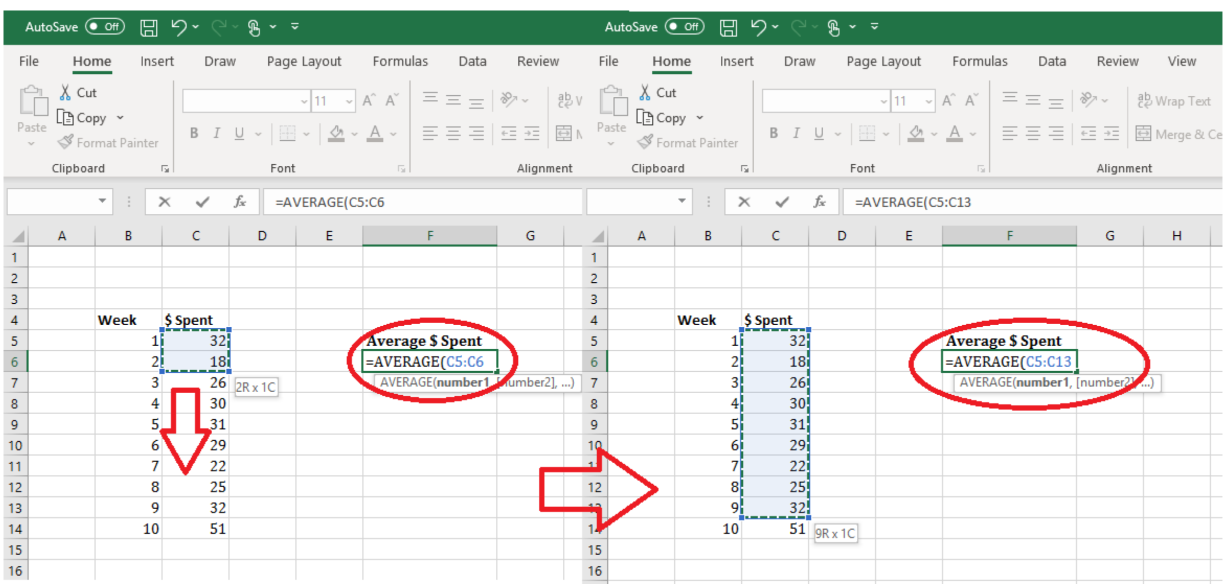
Division by clicking on specific cell values with a backslash for division.

Utilizing Equations in Spreadsheets

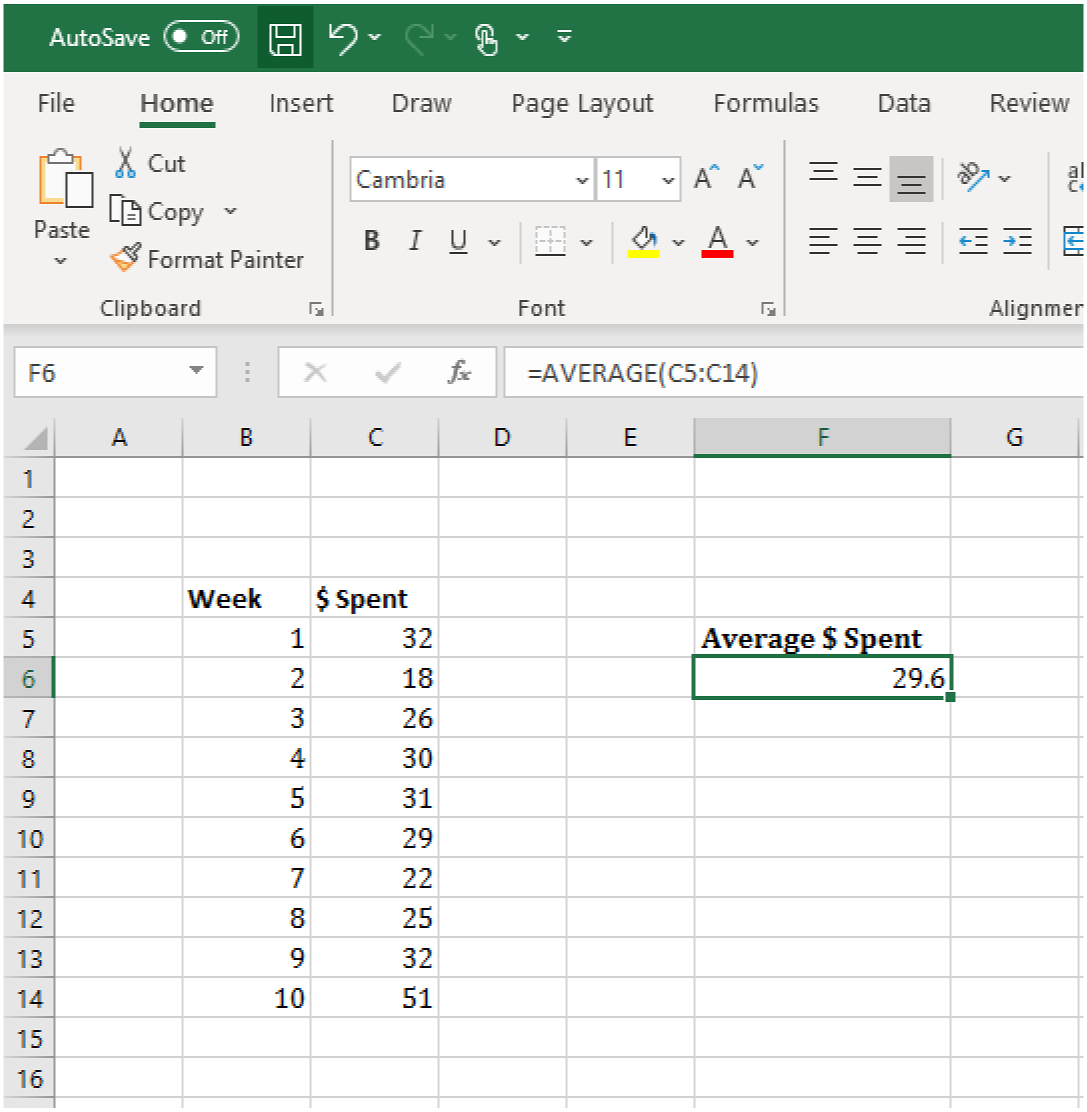
Sometimes, you will need to use an equation in spreadsheets that aren't simply adding/subtracting or multiplying/dividing. When needing to enter a specific equation, usually you can type the name of the equation into the cell after an equal sign and a list will populate for you to choose from.



Entering an equation into a cell is as easy as beginning to type the equaitons name and then selecting it from a narrowed down list.



Multiply data values, or cells, can be selected for an equation if you left click the mouse on one value and "drag" the highlighted box to include all the values you want to use.



When an equation you entered is complete, hitting enter key will terminate the equation and the spreadsheet will complette the calculation. As long as the equation is written correctly.

Key Concepts

- **The weighted average is calculated as**

$$\frac{\sum_{i=1}^{n}(x_i \cdot w_i)}{\sum_{i=1}^{n} w_i}$$

- **Mean or Expected Value:** $\mu = \sum_{x \in X} xP(x)$
- **The weighted moving average uses the weighted average equation and moves down a list of data one data value at a time.**
- **The weighted moving average can be used to "smooth out" data to visualize trends over time.**

Glossary

Expected Value
: expected arithmetic average when an experiment is repeated many times; also called the mean. Notations: μ. For a discrete random variable (RV) with probability distribution function $P(x)$,the definition can also be written in the form $\mu = \sum xP(x)$.

The Law of Large Numbers
: As the number of trials in a probability experiment increases, the difference between the theoretical probability of an event and the relative frequency probability approaches zero.

Weighted Moving Average
: A weighted average that moves through data one data value at a time to "smooth out" the data over time allowing for an easier identification of trends and habits of the data.

Ratios: Lessons 7.A - 7.F
By the end of this section, you will be able to:

- Use Pie charts/graphs
- Understand Part to part and whole Ratios
- Calculate Change in Value
- Write in Scientific Notation
- Understand Debt to Income Ratios

This correlates to Lessons 10A-B from Corequisite MAT 1043(QR) and NCBO.

Note:
In case you missed something in class, or just want to review a specific topic covered in this Module, here is a list of topics covered:

1. Pie Graphs [link]
2. Part-to-Whole Ratios[link]
3. Part-to-Part Ratios [link]
4. Working with Numbers in Scientific Notation [link]
5. Calculating Change in Values [link]
6. Debt-to-Income Ratios [link]
7. Key Concepts [link]

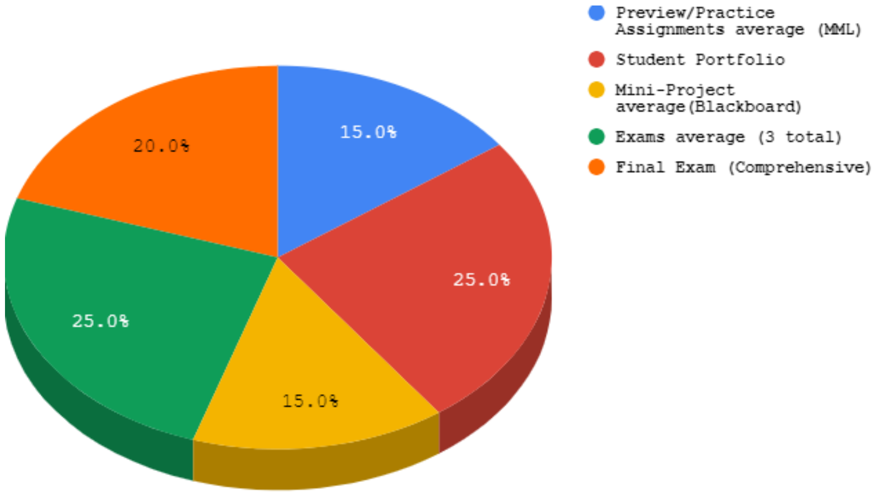
Pie Graphs [footnote]

This material was created by Amanda Towry using Google Sheets spreadsheet program.

Pie graphs can be used to represent parts or percentages of a whole. One of the most common applications of pie graphs are budgets or grade summaries like the one below.

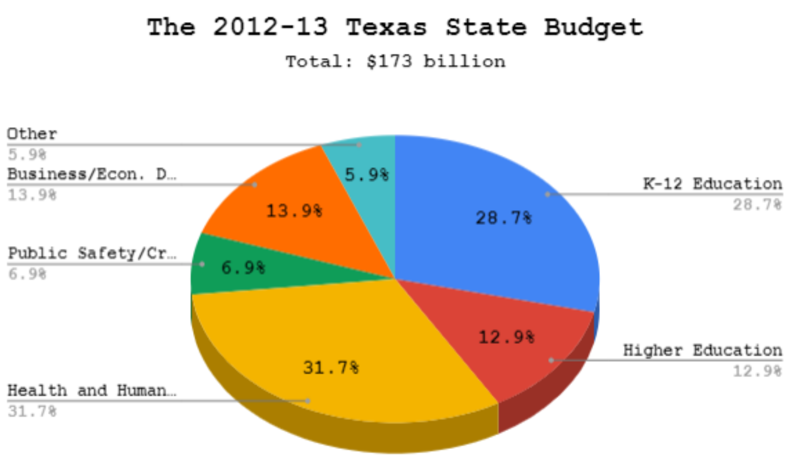
Preview/Practice Assignments Average (MML)	15%
Student Portfolio	25%
Mini-Project Average (Blackboard)	15%
Exams Average (3 Total)	25%
Final Exam (Comprehensive)	20%

Quantitative Reasoning Grade Details



Summary of grade break down for the Quatitative reasoning class you are taking right now.

Each percentage of a pie graph is determined by dividing the part of interest by the whole total. When out of a number other than percentages, the pie graph may look something like below:



Texas state All-Funds Budget for 2012-13. Each portion or percentage shown on the pie graph is of the total $173 billion.

Part-To-Whole Ratios [footnote]

This material was created by Amanda Towry using Google Sheets spreadsheet and Microsoft Word processing programs.

Remember that a ratio is a comparison of two numbers or quantities that are measured in the same unit.

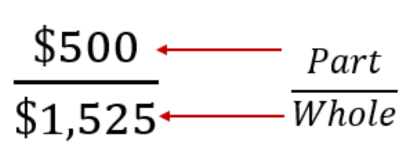
Part-to-whole ratios are simply a ratio where part or portion of something is compared to the whole.

Example:
Part to Whole Ratios
Below is a table showing the amounts that are budgeted for a student's monthly bills.

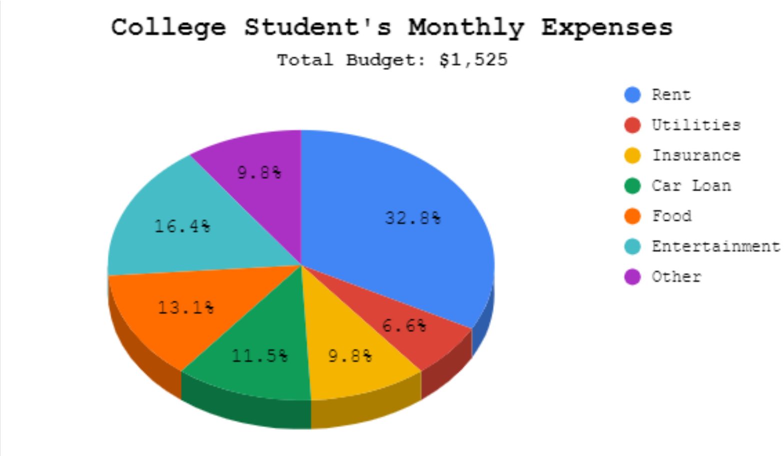
Expense	**Monthly Cost**
Rent	500
Utilities	100
Insurance	150
Car Loan	175
Food	200
Entertainment	250
Other	150
Total	**$1,525**

My Budget

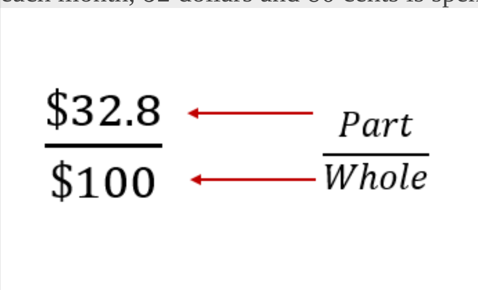
If we were to take the part if the budget planned for rent and compare it to the whole in a ratio, it would look like this:



Part to whole ratios allows us to see how a portion compares to a total. If we were to create a pie graph out of the monthly bills data, it would look like the following:



This chart tells us that the portion of the budget is 32% of the total money spent. For every 100 dollars spent each month, 32 dollars and 80 cents is spent on rent.



Part-To-Part Ratios [footnote]

This material was created by Amanda Towry using Microsoft Word processing program.

Part-to-part ratios are ratios where two separate parts of the same whole are compared to each other.

This makes it easier to compare the importance between two parts or to better understand how they each affect the total.

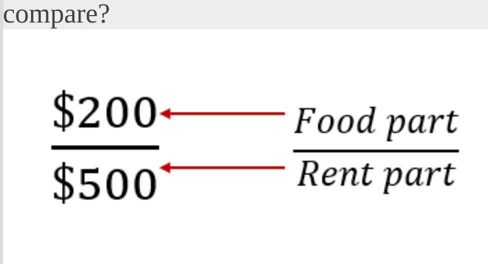
Example:
Part to Part Ratios
Below is the table from example 1 showing the amounts that are budgeted for a student's monthly bills.

Expense	Monthly Cost
Rent	500
Utilities	100

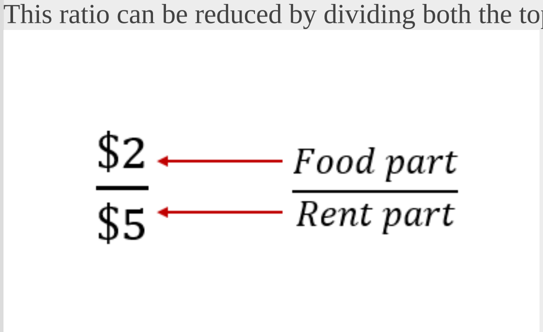
Insurance	150
Car Loan	175
Food	200
Entertainment	250
Other	150
Total	**$1,525**

My Budget

If we were to take the part of the budget reserved for food **and** the part reserved for rent, how would they compare?



This ratio can be reduced by dividing both the top and bottom by 100, giving the ratio:



This part to part ratio comparison tells us that for every 2 dollars spent on food, 5 dollars is spent on rent. This kind of part-to-part comparison lets us see that this college student **clearly** spends more money on rent than on food; because the rent part (the denominator) is larger than the food part (the numerator).

Working with Numbers in Scientific Notation[footnote]

Section material derived from Openstax Prealgebra: Polynomials-Integer Exponents and Scientific Notation

When working with **REALLY** large or small numbers, it is often easier for scientists to write them in **scientific notation**. **Scientific notation** is a number usually between 1 and 10 and multiplied by a factor of ten. This allows for fewer digits to be written down when expressing the numbers.

Equation:

Scientific Notation Format

$$a \times 10^n$$

where $a \geq 1$ and $a < 10$ and n is an integer.

Converting from decimal or a large number to scientific notation is pretty straight forward that only really requires the moving of the decimal point in the number. If the number doesn't have one (such as a whole integer like 5) you can imagine or write a decimal point at the end of the whole integer.

$4000. = 4 \times 10^3$	$0.004 = 4 \times 10^{-3}$
$4000. = 4 \times 10^3$	$0.004 = 4 \times 10^{-3}$
Moved the decimal point 3 places to the left.	Moved the decimal point 3 places to the right.

In both cases, the decimal was moved 3 places to get the first factor, 4, by itself.

- The power of 10 is positive when the number is larger than 1: $4000 = 4 \times 10^3$.
- The power of 10 is negative when the number is between 0 and 1: $0.004 = 4 \times 10^{-3}$.

Example:
Writing Numbers in Scientific Notation
Exercise:

Problem: Write $37{,}000$ in scientific notation.

Solution:
Solution

Step 1: Move the decimal point so that the first factor is greater than or equal to 1 but less than 10.	37000.
Step 2: Count the number of decimal places, n, that the decimal point was moved.	3.70000 4 places
Step 3: Write the number as a product with a power of 10.	3.7×10^4
If the original number is: • greater than 1, the power of 10 will be 10^n. • between 0 and 1, the power of 10 will be 10^{-n}	
Step 4: Check.	
10^4 is 10,000 and 10,000 times 3.7 will be 37,000.	
	$37{,}000 = 3.7 \times 10^4$

Example:
Writing Numbers in Scientific Notation
Exercise:

Problem: Write in scientific notation: 0.0052.

Solution:
Solution

	0.0052
Move the decimal point to get 5.2, a number between 1 and 10.	0.0052
Count the number of decimal places the point was moved.	3 places
Write as a product with a power of 10.	5.2×10^{-3}
Check your answer: 5.2×10^{-3} $5.2 \times \frac{1}{10^3}$ $5.2 \times \frac{1}{1000}$ 5.2×0.001 0.0052	
	$0.0052 = 5.2 \times 10^{-3}$

Note:

Move the decimal point so that the first factor is greater than or equal to 1 but less than 10.
Count the number of decimal places, n, that the decimal point was moved.
Write the number as a product with a power of 10.

- If the original number is:
  - greater than 1, the power of 10 will be 10^n.
  - between 0 and 1, the power of 10 will be 10^{-n}.

Check.

Converting from scientific notation number back to decimal notation is just as easy. The decimal is still moved the number of spaces identified by the exponent on the ten.

$$9.12 \times 10^4 = 91{,}200 \qquad 9.12 \times 10^{-4} = 0.000912$$

$$9.12\_\_ \times 10^4 = 91{,}200 \qquad \_\_\_9.12 \times 10^{-4} = 0.000912$$

In both cases the decimal point moved 4 places. When the exponent was positive, the decimal moved to the right. When the exponent was negative, the decimal point moved to the left.

Example:
Convert from Scientific Notation
Exercise:

Problem: Convert to decimal form: 6.2×10^3.

Solution:
Solution

Step 1: Determine the exponent, n, on the factor 10.	6.2×10^3
Step 2: Move the decimal point n places, adding zeros if needed.	6,200.
• If the exponent is positive, move the decimal point n places to the right. • If the exponent is negative, move the decimal point $\lvert n \rvert$ places to the left.	6,200
Step 3: Check to see if your answer makes sense.	
10^3 is 1000 and 1000 times 6.2 will be 6,200.	$6.2 \times 10^3 = 6{,}200$

Example:
Convert from Scientific Notation
Exercise:

Problem: Convert to decimal form: 8.9×10^{-2}.

Solution:
Solution

	8.9×10^{-2}
Determine the exponent n, on the factor 10.	The exponent is -2.
Move the decimal point 2 places to the left.	$\_\,8.9$
Add zeros as needed for placeholders.	0.089
	$8.9 \times 10^{-2} = 0.089$
The Check is left to you.	

Note:

Determine the exponent, n, on the factor 10.

Move the decimal n places, adding zeros if needed.

- If the exponent is positive, move the decimal point n places to the right.
- If the exponent is negative, move the decimal point $|n|$ places to the left.

Check.

Calculating Change In Values [footnote]

This material was created by Amanda Towry using Microsoft Word processing program.

Absolute Change
Absolute change describes the specific change in a quantity or value. It is the **difference** between a new and old value. It can be calculated as:

$$\textit{Absolute Change} = \textit{New value} - \textit{Old value}$$

When calculating the absolute change, two situations can occur:

- A **POSITIVE** difference is an **INCREASE** over time.
- A **NEGATIVE** difference is a **DECREASE** ovdr time.

Example:
Calculating Absolute Change
Exercise:

Problem:

A diversified stock portfolihas a worth that changes from $\$1{,}500$ to$\$2{,}250$. What is the absolute change of the stock portfolio value?

Solution:

We will subtract the old value from the new value. $\$2{,}250 - 1{,}500 = \$750$ Notice that the difference is positive. This means that the change is an increase. As a complete sentence, the answer would be: $\text{There was a \$750 increase in the stock portfolio value.}$

Example:
Calculating Absolute Change
Exercise:

Problem:

An animal shelter reports that their number of animals awaiting adoption changed from 82 to 46 animals over 6 months. How did the animal popualtion change over time?

Solution:

We will subtract the old value from the new value. $46 - 82 = -36$ Notice that the difference is negative. This means that the change is a decrease. As a complete sentence, the answer would be: $\text{There was a decrease of 36 animals in the animal shelter population over 6 months.}$

Relative Change
Relative change describes the ratio comparison of the absolute change to the original value. It can be calculated as:

$$\boldsymbol{Relative\ Change} = \frac{\boldsymbol{Absolute\ change}}{\boldsymbol{Old\ value}} = \frac{\boldsymbol{New\ value} - \boldsymbol{Old\ value}}{\boldsymbol{Old\ value}}$$

The two possible situations still apply to relative change as they did to absolute change:

- A **POSITIVE** difference is an **INCREASE** over time.
- A **NEGATIVE** difference is a **DECREASE** ovdr time.

Because the relative change is a ratio, we can multiply it by 100 and report it as a percentage.

Example:
Calculating Relative Change
Exercise:

Problem:

A diversified stock portfolio has a worth that changes from $\$1{,}500$ to$\$2{,}250$. What is the relative change of the stock portfolio value?

Solution:

We will subtract the old value from the new value. $\$2{,}250 - 1{,}500 = \$750$ Dividing the result by the old value gives: $\$750/\$1{,}500 = 0.5$.

To find the percent, we multiply this result by 100. $0.50*100 = 50\%$. The final answer as a complete sentence would be: $\text{The stock portfolio has increased in value by } 50\%$

Example:
Calculating Relative Change
Exercise:

Problem:

An animal shelter reports that their number of animals awaiting adoption changed from 82 to 46 animals over 6 months. How did the animal popualtion change over time?

Solution:

We will subtract the old value from the new value. $46 - 82 = -36$ Notice that the difference is negative. Dividing this outcome by the old value gives the result: $-36/46 = -0.783$

To find the percent, we multiply this result by 100. $-0.783*100 = -78.3\%$. The final answer as a complete sentence would be:
$\text{The number of animals awaiting adoption at the shelter has DECREASED by } 78.3\% \text{ over 6 months.}$

Debt-To-Income Ratios [footnote]

This material was created by Amanda Towry using Microsoft Word processing program.

One of the major decisions an individual faces when becoming an adult is to take on and manage debt. **Debt** is what we call when a favor, service, obligation, or money is owed to an individual or organization.

The most common forms of debt for college students are a student or a car loan. When a person is applying for private loans, credit cards, etc. banks will often look at what is called your debt-to-income ratio. **Debt-to-income ratio (DTI)** is a ratio that compares a person's monthly debt payments to their **gross monthly income**.

$$DTI = \frac{Monthly\ debt\ payments}{Gross\ monthly\ income}$$

Specifically, there are two subcategories of DTI we will look at.

Note: Front-End-DTI is a ratio of housing payments to total income and is used in reference to mortgages.

$$Front - End\ DTI = \frac{Housing\ payments}{Total\ Income}$$

Example:

Front End DTI

Congratulations! You have finished college and have been offered a good job that pays $2,100 per month.

Exercise:

Problem:

What should your maximum rent payment be to make sure that your front-end DTI is less than or equal to 28%?

Solution:

The DTI ratio we are looking for is 28% and so this is the value you pluig in for the left side of the equal sign. The monthly income is given as $2,100 which goees in the denominator of the right side of the DTI equation. This gives the following equation to solve:

$$28\% \geq \frac{Rent\ Payment}{\$2{,}100}$$

First, you need to convert the DTI from a percent to a decimal:

$$0.28 \geq \frac{Rent\ Payment}{\$2{,}100}$$

In order to solve for the unknown rent payment, both sides must be multiplied by the denominator, $2,100:

$$(0.28) * (\$2{,}100) \geq Rent\ Payment$$

Solving by multiplication gives us that the rent payment should be no more that $588

$$Rent\ Payment \leq \$588$$

Note: Back-End DTI is a ratio of all current and recurring debt payments as well as your mortgage or rent, to total income.

$$Back - End\ DTI = \frac{All\ recurring\ debt\ payments}{Total\ Income}$$

Example:
Back End DTI
After taking the job that pays $2,100 per month, you decide to tackle your student loan debt from college. You have built up a student loan debt of $18,500 and they carry a 3.5% interest rate.
Exercise:

Problem:

Your rent is $500 per month and you pay $150 per month for your brand new car loan. What should your maximum student loan payment be to maintain a back-end DTI of no more than 32%

Solution:

In the back-end DTI equation, all recurring debt payments are considered in the numerator. This means we must consider the money paid for rent, the car loan, and any potential loan payments. Because we are trying to figure out **what** that student loan payment should be, we can represent it with a variable.

$$38\% = \frac{\$500 + \$150 + x}{\$2{,}100}$$

Next, we will want to convert the 38% to a decimal.

$$0.38 = \frac{\$500 + \$150 + x}{\$2{,}100}$$

Next, we want to remove the denominator by multiplying both sides by 2,100.

$$(0.38)(\$2{,}100) = \$500 + \$150 + x$$

Multiplying out the math on the left side of the equal sign and combining like terms on the right results in an equation that can be simply solved by subtraction:

$$\$798 = \$650 + x$$

$$x = \$148$$

Remember that we were looking for the maximum loan payment possible and that is what the variable represents. This means that the loan payment maximum is $148 per month.

Key Concepts

- **Absolute change is**
  Equation:

$$\text{New value - Old value}$$

- **Relative change is**
  Equation:

$$\frac{\text{Absolute change}}{\text{Old value}} = \frac{\text{New value - Old value}}{\text{Old value}}$$

- **A number in scientific notation is of the format**
  Equation:

$$a \times 10^n$$

- **Debt to income ratio is represented as**
  Equation:

$$\text{DTI} = \frac{\text{Monthly debt payments}}{\text{Gross monthly income}}$$

- **Front-emd DTI is represented as**
  Equation:

$$\text{Front-end DTI} = \frac{\text{Housing payments}}{\text{Total income}}$$

- **Back-end DTI is represented as Equation:**

$$\text{Back-end DTI} = \frac{\text{All recurring debt payments}}{\text{Total income}}$$

- **Convert from Decimal Notation to Scientific Notation:** To convert a decimal to scientific notation:

  Move the decimal point so that the first factor is greater than or equal to 1 but less than 10.
  Count the number of decimal places, n, that the decimal point was moved.
  Write the number as a product with a power of 10.
    - If the original number is greater than 1, the power of 10 will be 10^n.
    - If the original number is between 0 and 1, the power of 10 will be 10^n.

  Check.

- **Convert Scientific Notation to Decimal Form:** To convert scientific notation to decimal form:

  Determine the exponent, n, on the factor 10.
  Move the decimal n places, adding zeros if needed.
    - If the exponent is positive, move the decimal point n places to the right.
    - If the exponent is negative, move the decimal point $|n|$ places to the left.

  Check.

Glossary

Back-End DTI
: A type of DTI that compares all of your recurring debt payments, including rent or mortgage, to your gross monthly income.

Debt-to-Income Ratio (DTI)
: A ratio used in personal finances that compares your monthly debt payments to your gross monthly income.

Front-End DTI
: A type of DTI that compares your housing payments, such as rent or mortgage, to your gross monthly income.

Part-to-Whole Ratio
: A ratio that compares a part or some of the parts of something to the whole.

Part-to-Part Ratio
: A ratio that compares a part of a whole to another part of the same whole.

Linear & Proportional Relationships: Lessons 8.A-8.C
This corresponds to Lessons 8A-E from Corequisite MAT 1043(QR) and NCBO courses.

Note:
In case you missed something in class, or just want to review a specific topic covered in this Module, here is a list of topics covered:

Representations of Mathematical Data [footnote]

This material was created by Amanda Towry using Microsoft Word processing program.

Often times when mathematical relationships or data needs to be shared or presented, one of the most important decisions you will make will be on how you present your data. In mathematics, and in your Quantitative Reasoning course, there are four ways that data and information can and will be presented for interpretation.

Verbal Representations
A **verbal representation** is one where words are used to describe a mathematical problem, situation, or data set.

Example: Verbal Representation
Mary has 5 red marbles, 3 blue marbles, and 7 green marbles. If Mary choose two marble randomly from her collection out of a bag, what is the probability she will choose a red **and** a green one?

Numerical Representations
A **numerical representation** (sometimes called tabular) is one in which numbers describe or represent a mathematical problem, situation, or data set. It is most common to see a table for this type of representation.

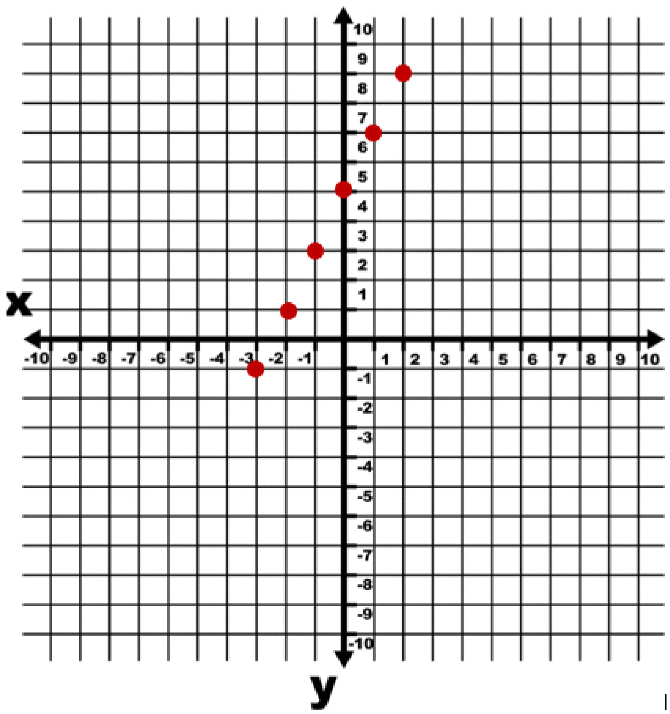
Example: Numerical Representation

x	y
-3	-1
-2	1
-1	3
0	5
1	7
2	9
3	11

Numerical Representation of a linear function

Graphical Representation

A **graphical representation** is one in which mathematical information is plotted and generates a graph representing some relationship.



Graphical representation of data table values that matches the numerical representation example.

Symbolic/Algebraic Representation

A **symbolic/algebraic representation** is one in which mathematical information or relationships are presented using numbers and variables such as in an algebraic equation. A **graphical representation** is one in which mathematical information is plotted and generates a graph representing some relationship.

$$y = 2 * x + 5$$

Symbolic representation of data table values that matches the graphical representation example.

Rates of Change [footnote]

This material was created by Amanda Towry using Microsoft Word processing program.

A **rate of change** is a relationship or unit rate that shows how one variable or quantity changes with another.

Hours Spent Working	Batches of Cookies Made
0	0
1	6
2	12
3	18
4	24
5	30

The most common way to investigate rates of change is in numerical or tabular form of data. How the values of x change compared to how the values of y change is called the rate of change. From hour 1 to hour 2, 6 batches of cookies were made. From hour 2 to hour 3, 6 more batches were made. This relationship of change is often represented with a ratio where the dependent variable is in the numerator and the independent variable is in the denominator. The rate of change here would be:

$$\frac{6\ batches\ of\ cookies}{1\ hour\ spent\ working}$$

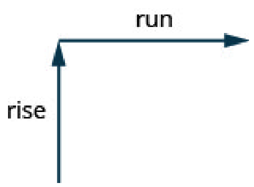
Slope as A Rate of Change [footnote]

Section material derived from Openstax Introductory Statistics: The Linear Regression and Correlation-Linear Equations and Prealgebra: Graphs-Understand Slope of a Line

In mathematics, the measure of the rate of change of a line is called the **slope** of the line.

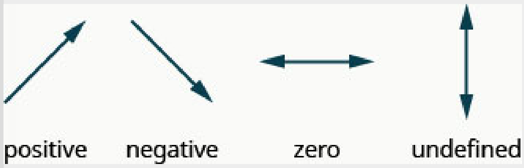
The concept of slope has many applications in the real world. In construction, the pitch of a roof, the slant of the plumbing pipes, and the steepness of the stairs are all applications of slope. And as you ski or jog down a hill, you definitely experience slope.

We can assign a numerical value to the slope of a line by finding the ratio of the rise and run. The rise is the amount the vertical distance changes while the run measures the horizontal change, as shown in this illustration. The slope is a rate of change. See [link].



Note:
The slope of a line is $m=\frac{\text{rise}}{\text{run}}$.
The rise measures the vertical change and the run measures the horizontal change. Creating a ratio out of these values gives us the rate of change of the line.

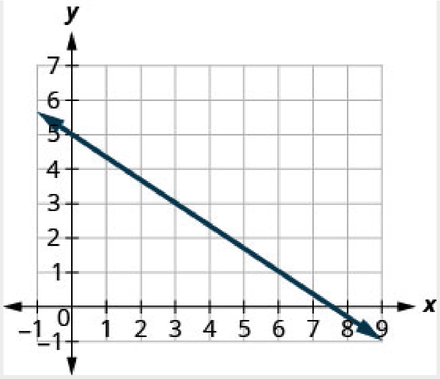
Note:



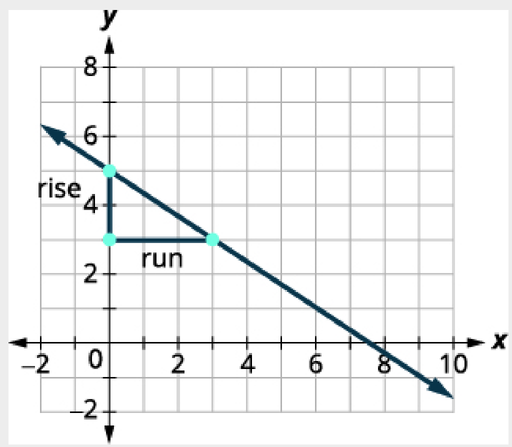
There are multiple methods to find the slope of a line depending on the type of representation presented.

Example:
Finding the Slope of a Line
Exercise:

Problem: Find the slope of the line shown.



Solution:

<table>
<tr><td>Locate two points on the graph whose coordinates are integers.</td><td>$(0, 5)$ and $(3, 3)$</td></tr>
<tr><td>Starting at $(0, 5)$, sketch a right triangle to
$(3, 3)$ as shown in this graph.</td><td>
</td></tr>
<tr><td>Count the rise— since it goes down, it is negative.</td><td>The rise is -2.</td></tr>
<tr><td>Count the run.</td><td>The run is 3.</td></tr>
<tr><td>Use the slope formula.</td><td>$m = \frac{\text{rise}}{\text{run}}$</td></tr>
<tr><td>Substitute the values of the rise and run.</td><td>$m = \frac{-2}{3}$</td></tr>
<tr><td>Simplify.</td><td>$m = -\frac{2}{3}$</td></tr>
</table>

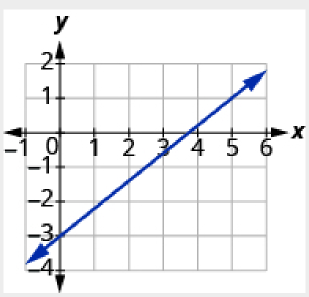
	The slope of the line is $-\frac{2}{3}$.
	So y decreases by 2 units as x increases by 3 units.

Find the slope of a line from its graph using $m = \frac{\text{rise}}{\text{run}}$.

Locate two points on the line whose coordinates are integers.
Starting with one point, sketch a right triangle, going from the first point to the second point.
Count the rise and the run on the legs of the triangle.
Take the ratio of rise to run to find the slope:$m = \frac{\text{rise}}{\text{run}}$.

Example:
Finding slope with an integer y-intercept
Exercise:

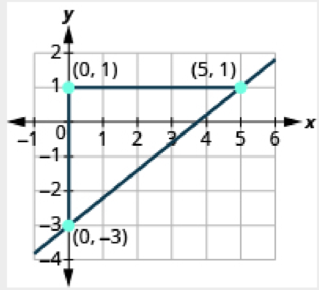
Problem: Find the slope of the line shown:



Solution:
Solution

Locate two points on the graph, choosing points whose coordinates are integers. We will use $(0, -3)$ and $(5, 1)$.

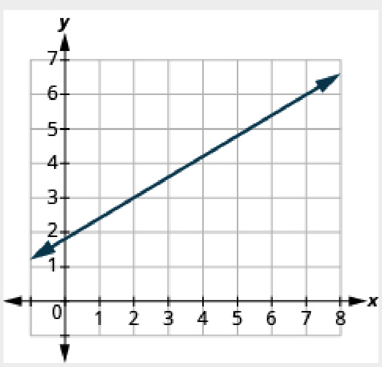
Starting with the point on the left, $(0, -3)$, sketch a right triangle, going from the first point to the second point, $(5, 1)$.

	
Count the rise on the vertical leg of the triangle.	The rise is 4 units.
Count the run on the horizontal leg.	The run is 5 units.
Use the slope formula.	$m = \frac{\text{rise}}{\text{run}}$
Substitute the values of the rise and run.	$m = \frac{4}{5}$
	The slope of the line is $\frac{4}{5}$.

Notice that the slope is positive since the line slants upward from left to right.

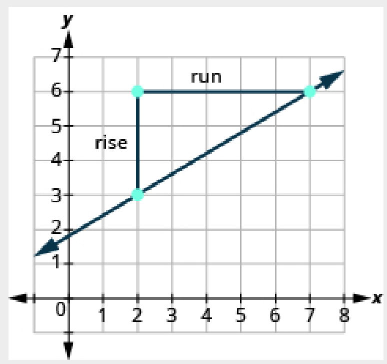
Example:
Finding slope without an integer y-intercept
Exercise:

Problem: Find the slope of the line shown:



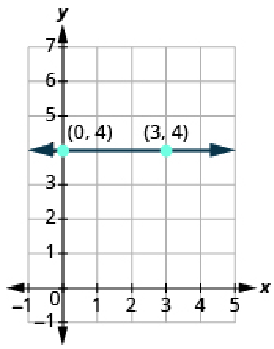
Solution:
Solution

Locate two points on the graph whose coordinates are integers.	$(2,3)$ and $(7,6)$
Which point is on the left?	$(2,3)$
Starting at $(2,3)$, sketch a right angle to $(7,6)$ as shown below.	

	
Count the rise.	The rise is 3.
Count the run.	The run is 5.
Use the slope formula.	$m=\frac{\text{rise}}{\text{run}}$
Substitute the values of the rise and run.	$m=\frac{3}{5}$
	The slope of the line is $\frac{3}{5}$.

Some lines are not at an angle but are either completely vertical or horizontal. These are special kinds of slope because the numerical equations will only ever have one variable; either just the x or y.

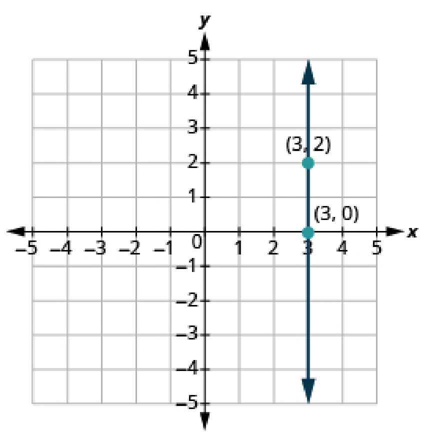
The Slope of a Horizontal Line
The slope of a horizontal line, $y=b$, is 0.



What is the rise?	The rise is 0.
What is the run?	The run is 3.
What is the slope?	$m = \frac{\text{rise}}{\text{run}}$
	$m = \frac{0}{3}$
	$m = 0$

All horizontal lines have slope 0. When the y-coordinates are the same, the rise is 0.

The Slope of a Vertical Line
The slope of a vertical line, $x = a$, is undefined.



What is the rise?	The rise is 2.
What is the run?	The run is 0.
What is the slope?	$m = \frac{\text{rise}}{\text{run}}$
	$m = \frac{2}{0}$

All vertical lines have slope undefined. When the x-coordinates are the same, the rise is undefined. This is because any number minus itself is zero and you cannot divide by 0.

The Slope From Two Points

The slope of the line between two points (x_1, y_1) and (x_2, y_2) is:

Equation:

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

OR

Equation:

$$\begin{array}{c} y \text{ of the second point minus } y \text{ of the first point} \\ \text{over} \\ x \text{ of the second point minus } x \text{ of the first point.} \end{array}$$

Example:

Finding the Slope of a Line with Two Points

Exercise:

Problem:

Use the slope formula to find the slope of the line through the points $(-2, -3)$ and $(-7, 4)$.

Solution:

We'll call $(-2,-3)$ point #1 and $(-7,4)$ point #2.

$$\begin{pmatrix} x_1, & y_1 \\ -2, & -3 \end{pmatrix} \begin{pmatrix} x_2, & y_2 \\ -7, & 4 \end{pmatrix}$$

Use the slope formula.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Substitute the values.

y of the second point minus y of the first point

x of the second point minus x of the first point

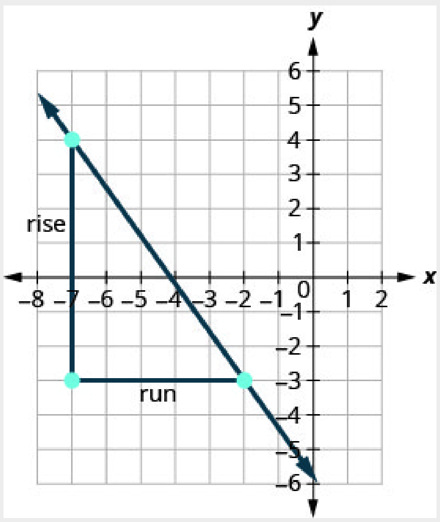
$$m = \frac{4-(-3)}{-7-(-2)}$$

Simplify.

$$m = \frac{7}{-5}$$

$$m = -\frac{7}{5}$$

Let's verify this slope on the graph shown.



Equation:

$$m = \frac{\text{rise}}{\text{run}}$$

$$m = \frac{7}{-5}$$

$$m = -\frac{7}{5}$$

Proportional Relationships [footnote]

This material was created by Amanda Towry using Microsoft Word processing program.

Relationships are considered proportional relationships when the ratio between related sets of data values is constant. To know if a relationship is proportional, look for the following aspects:

- **Algebraic/symbolic representations** (or equations) will tell you if they are proportional when you look at their y-intercept, or the b.

- **Proportional relationships** will not have a y-intercept, or will not have a b value, or the b will be 0 .
- **Non-Proportional** relationships will have a non-zero y-intercept.

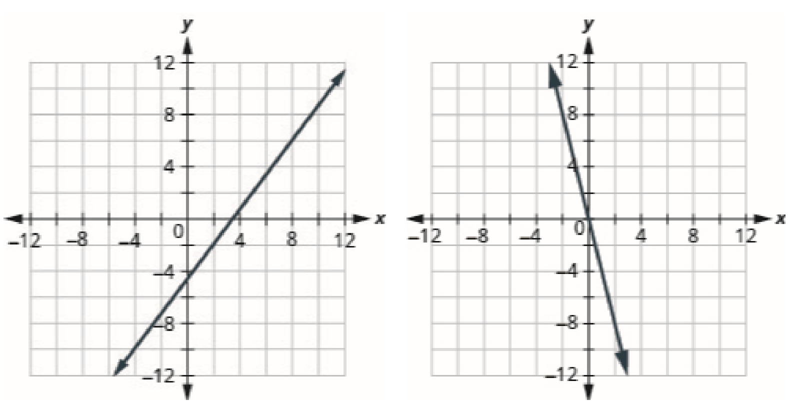
- **Numerical/tabular representations** will have the coordinate pair $(0, 0)$ in the table.

x	y
-4	-10
1	1
8	6

x	y
-1	8
0	0
1	-8

On the left is an example of a non-proportional numerical/tabular representation and on the right is an example of a proportional numerical/tabular representation.

- **Graphical representations** will show one of the following:
  - Pass through the origin if it is **proportional**
  - Not pass trhough the origin if it is **non-proportional**.



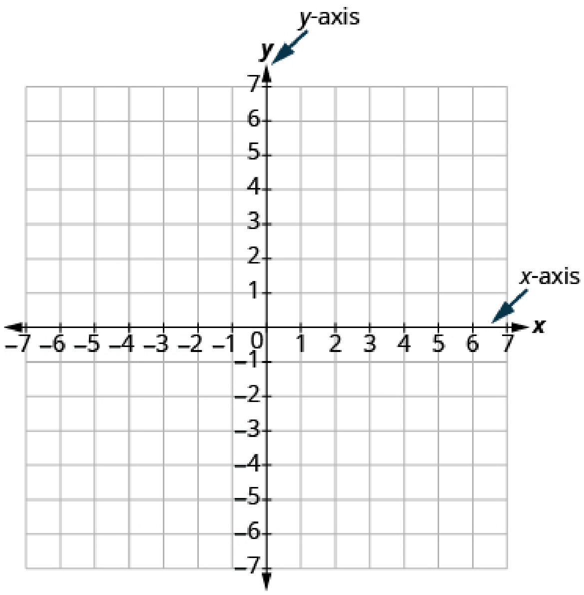
On the left is an example of a non-proportional graphical representation and on the right is an example of a proportional graphical representation.

Note:
All proportional relationships are linear but not all linear relationships are proportional
.

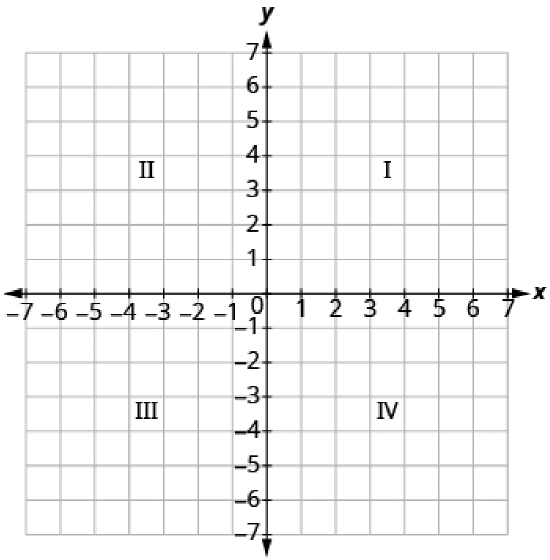
The Rectangular Coordinate System [footnote]

Section material derived from Openstax Prealgebra: The Properties of Real Numbers-Use the Rectangular Coordinate System

The rectangular coordinate system is also called the x-y plane, the coordinate plane, or the Cartesian coordinate system (since it was developed by a mathematician named René Descartes.)



The x-axis and the y-axis form the rectangular coordinate system. These axes divide a plane into four areas, called **quadrants**. The quadrants are identified by Roman numerals, beginning on the upper right and proceeding counterclockwise. See [link].

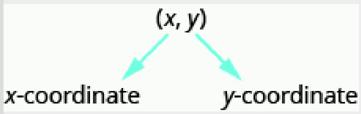


In the rectangular coordinate system, every point is represented by an **ordered pair**. The first number in the ordered pair is the *x*-coordinate of the point, and the second number is the *y*-coordinate of the point.

Note: The point $(0, 0)$ is called the **origin**. It is the point where the *x*-axis and *y*-axis intersect.

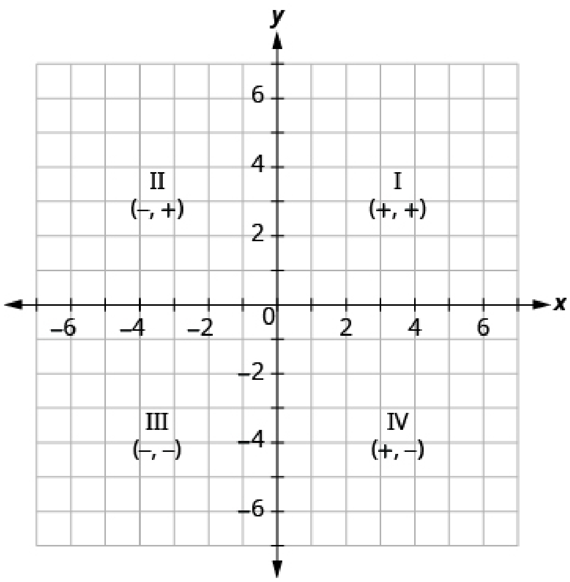
Note: (x, y) gives the coordinates of a point in a rectangular coordinate system.
Equation:

$$\begin{array}{l}\text{The first number is the } x\text{-coordinate.}\\ \text{The second number is the } y\text{-coordinate.}\end{array}$$



We can summarize sign patterns of the quadrants as follows.

Quadrant I	Quadrant II	Quadrant III	Quadrant IV
(*x*,*y*)	(*x*,*y*)	(*x*,*y*)	(*x*,*y*)
(+,+)	(−,+)	(−,−)	(+,−)

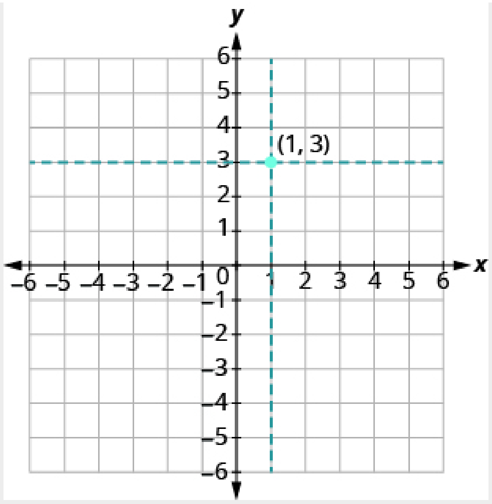


Example:
Plotting Points in the Rectangular Coordinate System
Exercise:

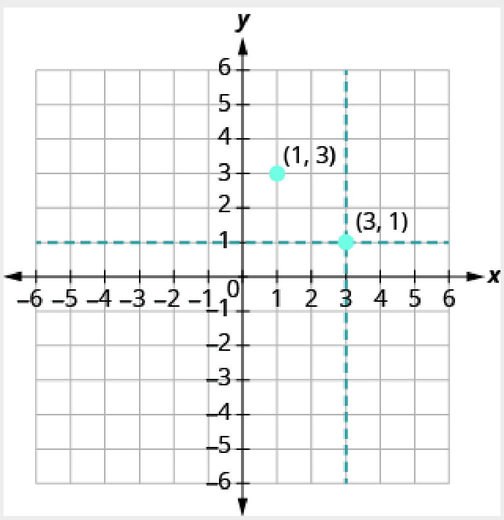
Problem: Plot $(1,3)$ and $(3,1)$ in the same rectangular coordinate system.

Solution:
Solution

The coordinate values are the same for both points, but the x and y values are reversed. Let's begin with point $(1,3)$. The x-coordinate is 1 so find 1 on the x-axis and sketch a vertical line through $x=1$. The y-coordinate is 3 so we find 3 on the y-axis and sketch a horizontal line through $y=3$. Where the two lines meet, we plot the point $(1,3)$.



To plot the point $(3, 1)$, we start by locating 3 on the x-axis and sketch a vertical line through $x = 3$. Then we find 1 on the y-axis and sketch a horizontal line through $y = 1$. Where the two lines meet, we plot the point $(3, 1)$.



Notice that the order of the coordinates does matter, so, $(1, 3)$ is not the same point as $(3, 1)$.

Graph a Linear Equation by Plotting Points [footnote]

Section material derived from Openstax Prealgebra: Graphs-Understand Slope of a Line

Let's graph the equation $y = 2x + 1$ by plotting points.

We start by finding three points that are solutions to the equation. We can choose any value for x or y, and then solve for the other variable.

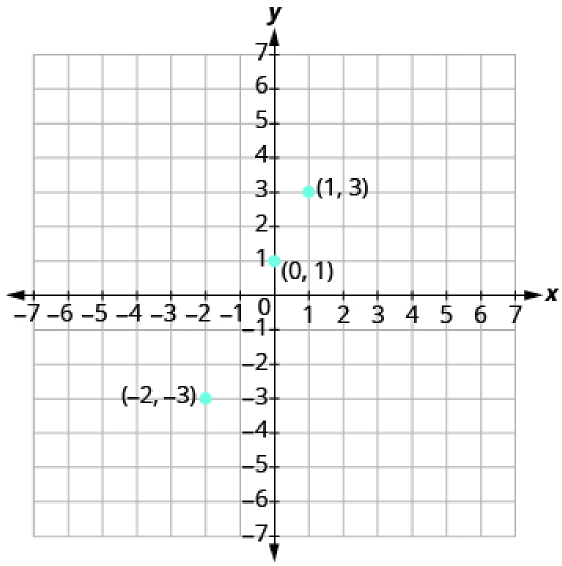
Since y is isolated on the left side of the equation, it is easier to choose values for x. We will use $0, 1$, and -2 for x for this example. We substitute each value of x into the equation and solve for y.

$x = -2$	$x = 0$	$x = 1$
$y = 2x + 1$	$y = 2x + 1$	$y = 2x + 1$
$y = 2(-2) + 1$	$y = 2(0) + 1$	$y = 2(1) + 1$
$y = -4 + 1$	$y = 0 + 1$	$y = 2 + 1$
$y = -3$	$y = 1$	$y = 3$
$(-2, -3)$	$(0, 1)$	$(1, 3)$

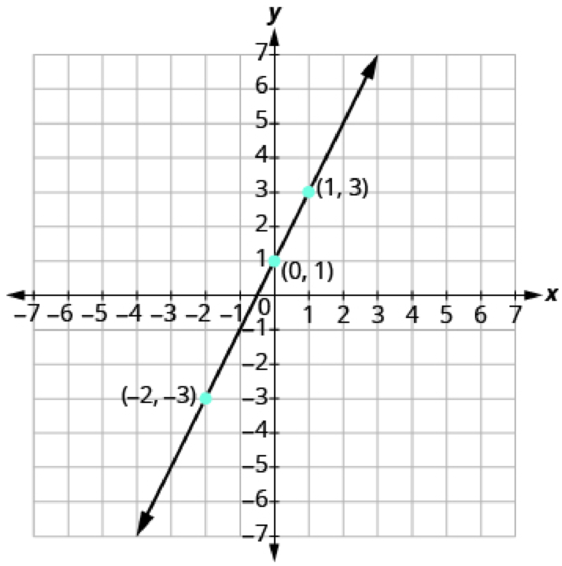
We can organize the solutions in a table.

$y = 2x + 1$		
x	y	(x, y)
0	1	$(0, 1)$
1	3	$(1, 3)$
-2	-3	$(-2, -3)$

Now we plot the points on a rectangular coordinate system. Check that the points line up. If they did not line up, it would mean we made a mistake and should double-check all our work.



Draw the line through the three points. Extend the line to fill the grid and put arrows on both ends of the line. The line is the graph of $y = 2x + 1$.



Note:

Find three points whose coordinates are solutions to the equation. Organize them in a table.
Plot the points on a rectangular coordinate system. Check that the points line up. If they do not, carefully check your work.
Draw the line through the points. Extend the line to fill the grid and put arrows on both ends of the line.

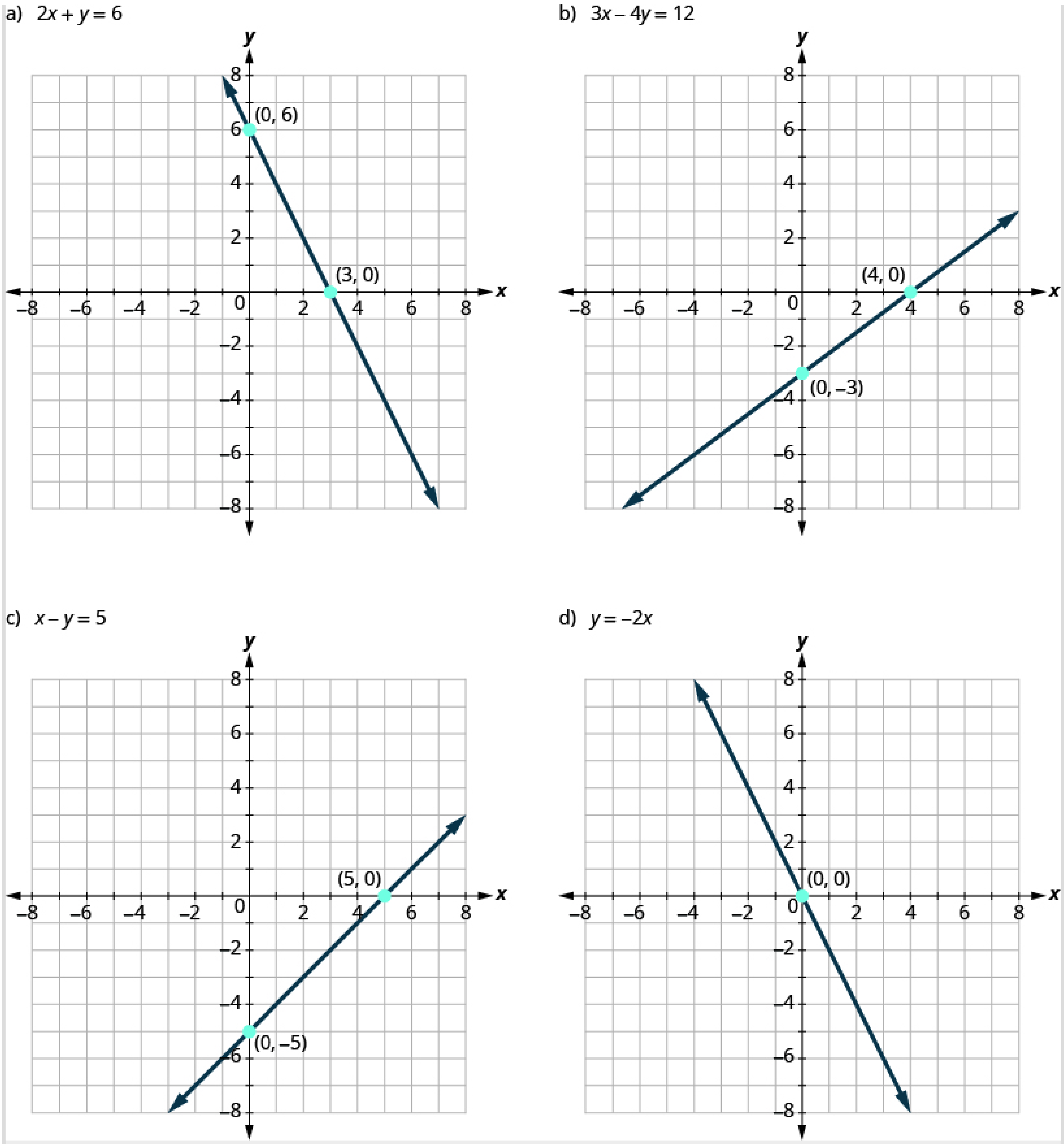
Graphing a Linear Equation with Intercepts [footnote]

Section material derived from Openstax Prealgebra: Graphs-Understand Slope of a Line

Every linear equation has a unique line that represents all the solutions of the equation. When graphing a line by plotting points, each person who graphs the line can choose any three points, so two people graphing the line might use different sets of points.

At first glance, their two lines might appear different since they would have different points labeled. But if all the work was done correctly, the lines will be exactly the same line. One way to recognize that they are indeed the same line is to focus on where the line crosses the axes. Each of these points is called an **intercept of the line**.

Note:
Each of the points at which a line crosses the x-axis and the y-axis is called an intercept of the line.



First, notice where each of these lines crosses the *x*- axis:

Figure:	**The line crosses the x-axis at:**	**Ordered pair of this point**

Figure:	The line crosses the x-axis at:	Ordered pair of this point
42	3	(3,0)
43	4	(4,0)
44	5	(5,0)
45	0	(0,0)

For each row, the *y*- coordinate of the point where the line crosses the *x*- axis is zero. The point where the line crosses the *x*- axis has the form $(a, 0)$; and is called the *x-intercept* of the line. The **x-** intercept occurs when y is zero.

Now, let's look at the points where these lines cross the y-axis.

Figure:	The line crosses the y-axis at:	Ordered pair for this point
42	6	(0,6)
43	-3	(0,-3)
44	-5	(0,-5)
45	0	(0,0)

For each row, the *x*- coordinate of the point where the line crosses the *y*- axis is zero. The point where the line crosses the *y*- axis has the form $(0, b)$; and is called the *y-intercept* of the line. The **y-** intercept occurs when x is zero.

Note:
The x-intercept is the point, $(a, 0\ )$, where the graph crosses the x-axis. The x-intercept occurs when y is zero.
The y-intercept is the point, $(0, b\ )$, where the graph crosses the y-axis.
The y-intercept occurs when x is zero.

To graph a linear equation by plotting points, you can use the intercepts as two of your three points. Find the two intercepts, and then a third point to ensure accuracy, and draw the line. This

method is often the quickest way to graph a line.

Example:
Graphing with Intercepts
Exercise:

Problem: Graph $-x + 2y = 6$ using intercepts.

Solution:
Solution

First, find the x-intercept. Let $y = 0$,

$$\begin{aligned} -x + 2y &= 6 \\ -x + 2(0) &= 6 \\ -x &= 6 \\ x &= -6 \end{aligned}$$

The x-intercept is $(-6, 0)$.

Now find the y-intercept. Let $x = 0$.

$$\begin{aligned} -x + 2y &= 6 \\ -0 + 2y &= 6 \end{aligned}$$

$$\begin{aligned} 2y &= 6 \\ y &= 3 \end{aligned}$$

The y-intercept is $(0, 3)$.

Find a third point. We'll use $x = 2$,

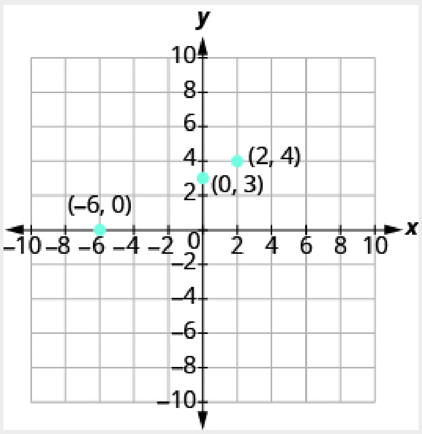
$$\begin{aligned} -x + 2y &= 6 \\ -2 + 2y &= 6 \end{aligned}$$

$$\begin{aligned} 2y &= 8 \\ y &= 4 \end{aligned}$$

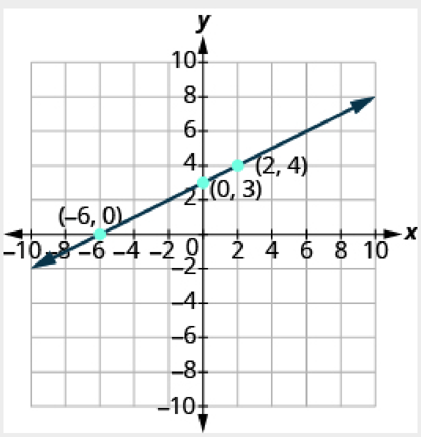
A third solution to the equation is $(2, 4)$.

Summarize the three points in a table and then plot them on a graph.

$-x+2y=6$		
x	***y***	***(x,y)***
-6	0	$(-6,0)$
0	3	$(0,3)$
2	4	$(2,4)$



Do the points line up? Yes, so draw line through the points.



Note:

Find the $x-$ and y-intercepts of the line.

- Let $y=0$ and solve for x
- Let $x=0$ and solve for y.

Find a third solution to the equation.
Plot the three points and then check that they line up.
Draw the line.

Key Concepts

- **Intercepts**
  - The *x*-intercept is the point, $(a, 0)$, where the graph crosses the *x*-axis. The *x*-intercept occurs when y is zero.
  - The *y*-intercept is the point, $(0, b)$, where the graph crosses the *y*-axis. The *y*-intercept occurs when y is zero.
  - The *x*-intercept occurs when y is zero.
  - The *y*-intercept occurs when x is zero.
- **Find the *x* and *y* intercepts from the equation of a line**
  - To find the *x*-intercept of the line, let $y = 0$ and solve for *x*.
  - To find the *y*-intercept of the line, let $x = 0$ and solve for *y*.

x	**y**
	0
0	

- **Graph a line using the intercepts**

  Find the*x*-and*y*-intercepts of the line.
    - Let $y = 0$ and solve for *x*.
    - Let $x = 0$ and solve for *y*.

  Find a third solution to the equation.
  Plot the three points and then check that they line up.
  Draw the line.
- **Simple interest is represented and calculated with the equation**

$$I = P \cdot r \cdot t$$

- **Compound interest is represented and calculated with the equation**

$$B = P(r + 1)^t$$

Glossary

Graphical Representation
: A mathematical representation where the data or information is presented on a graph or coordinate place representing some relationship.

Intercepts of a Line
: Each of the points at which a line crosses the *x*-axis and the *y*-axis is called an intercept of the line.

Numerical Representation
: A mathematical representation where the data or information is presented in numerical form such as in a data table.

Proportional Relationship
: A mathematical relationship where the ratio between related sets of data values is constant.

Rate of Change
: A relationship or unit rate that shows how one variable or quatity changes with another.

Slope of a Line
: The slope of a line is $m = \frac{\text{rise}}{\text{run}}$. The rise measures the vertical change and the run measures the horizontal change.

Symbolic Representation
: Also known as algebraic representation, is a mathematical representation where where information or relationships are presented using numbers and symbols such as in an algebraic equation.

Verbal Representation
: A mathematical representation where words are used to describe the problem, situation, or data set.

Modeling Linear Relationships: 8.C -9.A

Note:
In case you missed something in class, or just want to review a specific topic covered in this Module, here is a list of topics covered:

1. Simple and Compound Interest [link]
2. Graphing Piecewise Functions [link]
3. Recognizing Interpolation or Extrapolation [link]
4. Key Concepts [link]

Simple and Compound Interest [footnote]

Section material derived by Amanda Towry

If you deposit money into a bank account, you are effectively lending money to the bank - and you can expect to receive interest in return. Similarly, if you borrow money from a bank (or from a department store, or a car dealership, for example) then you can expect to have to pay interest on the loan. That is the price of borrowing money.

If it stayed in your wallet, you could spend it any time you wanted. If the bank looked after it for you, then they could spend it, with the plan of making profit from it. The bank usually “pays" you to deposit it into an account, as a way of encouraging you to bank it with them, This payment is like a reward, which provides you with a reason to leave it with the bank for a while, rather than keeping the money in your wallet.

We call this reward "interest".

Simple Interest
Simple interest is where money is earned based off the **initial investment**. Simple interest can be calculated by the equation

$$I = P \cdot r \cdot t$$

I is interest earned, P is initial investment (or your initial deposit), r is the interest rate (usually substituted in as a decimal), and t is the time in years.

Example:
Simple Interest

You invest $500 in an account that earns 3% per year. What would be your account balance after 15 years?

You would first need to substitute each of the known values into the equation. Your initial investment is $500 and your time is 15 years. The interest rate is 3% but we must enter it in the equation as a decimal which is 0.03.

$$I = P \cdot r \cdot t$$

$$I = (\$500)(0.03)(15)$$

$$I = \$225$$

Now, the answer we have calculated is the amount of **interest** that was earned; not the total account balance after 15 years. In order to find the account balance, you must add the calculated interest to the initial deposit of $500.

$$\boldsymbol{Total\ Account\ Balance = \$500 + \$225 = \$725}$$

Example:
Simple Interest
Exercise:

Problem:

Find the rate if a principal of $\$8{,}200$ earned $\$3{,}772$ interest in 4 years.

Solution:
Solution

Organize the given information.

$$\begin{aligned} I &= \$3{,}772 \\ P &= \$8{,}200 \\ r &= ? \\ t &= 4 \text{ years} \end{aligned}$$

We will use the simple interest formula to find the rate.

Write the formula.	$I = Prt$
Substitute the given information.	$3{,}772 = 8{,}200r(4)$
Multiply.	$3{,}772 = 32{,}800r$
Divide.	$\frac{3{,}772}{32{,}800} = \frac{32{,}800r}{32{,}800}$
Simplify.	$0.115 = r$
Write as a percent.	$11.5\% = r$
Check your answer. Is 11.5% a reasonable rate if $3,772 was earned in 4 years?	
$I = Prt$	
$3{,}772 \stackrel{?}{=} 8{,}200(0.115)(4)$	
$3{,}772 = 3{,}772\checkmark$	
Write a complete sentence that answers the question.	The rate was 11.5%.

Example:
Simple Interest
Exercise:

Problem:

Eduardo noticed that his new car loan papers stated that with an interest rate of 7.5%, he would pay $\$6{,}596.25$ in interest over 5 years. How much did he borrow to pay for his car?

Solution:
Solution

We are asked to find the principal, P.

Organize the given information.

$$\begin{aligned} I &= 6{,}596.25 \\ P &= ? \\ r &= 7.5\% \\ t &= 5 \text{ years} \end{aligned}$$

Write the formula.	$I = Prt$
Substitute the given information.	$6{,}596.25 = P(0.075)(5)$
Multiply.	$6{,}596.25 = 0.375P$
Divide.	$\frac{6{,}596.25}{0.375} = \frac{0.375P}{0.375}$
Simplify.	$17{,}590 = P$
Check your answer. Is \$17,590 a reasonable amount to borrow to buy a car?	

$I = Prt$	
$6{,}596.25 \stackrel{?}{=} (17{,}590)(0.075)(5)$	
$6{,}596.25 = 6{,}596.25 \checkmark$	
Write a complete sentence that answers the question.	The amount borrowed was $17,590.

Compound Interest

Compound interest is when you earn interest based off a **previous year's balance**. You are essentially earning interest off the investment **and** any accumulated interest. This is an important difference between the simple interest formula and the compound interest formula because it means that in the compound interest formula a non-linear, or exponential, form is used to calculate it.

$$\boldsymbol{B = P(r + 1)^t}$$

We can see this equation develop from the simple interest equation by working through the concept step by step. Let's say we invest an initial balance of P into an account that pays interest i; then after the first year, the balance would be:

Equation:

$$\text{Closing Balance after 1 year} = \mathrm{P}(1 + \mathrm{i})$$

This is the same as Simple Interest because it only covers a single year. Then, if we take that out and re-invest it for another year - just as you saw us doing in the worked example above - then the balance after the second year will be:

Equation:

$$\begin{aligned}\text{Closing Balance after } 2 \text{ years} &= [P(1+i)] \times (1+i) \\ &= P(1+i)^2\end{aligned}$$

This is because you will earn interest off of the balance you invested for year two which just so happens to be the initial investment **plus** the interest you made off of it in year one! If you continue this process, you will have an exponential continuation of the account balance. This is why it is raised to the power of the number of years you invest it for.

Example:

Compound Interest

You invest $\$500$ in an account that earns 3% **compounded** every year. What would be your account balance after 15 years?

$$\boldsymbol{B = P(1+r)^t}$$
$$\boldsymbol{B = (\$500)(1+3\%)^{15}}$$
$$\boldsymbol{B = (\$500)(1+0.03)^{15}}$$
$$\boldsymbol{B = (\$500)(1.03)^{15}}$$
$$\boldsymbol{B = \$778.98}$$

Graphing Piecewise-Defined Functions [footnote]

Section material derived from Openstax College Algebra: Introductions to Functions-Domain and Range

Sometimes, we come across a function that requires more than one formula in order to obtain the given output. For example, in the toolkit functions, we introduced the absolute value function $f(x) = |x|$. With a domain of all real numbers and a range of values greater than or equal to 0, absolute value can be defined as the magnitude, or modulus, of a real number value regardless of sign. It is the distance from 0 on the number line. All of these definitions require the output to be greater than or equal to 0.

If we input 0, or a positive value, the output is the same as the input.
Equation:

$$f(x) = x \text{ if } x \geq 0$$

If we input a negative value, the output is the opposite of the input.
Equation:

$$f(x) = -x \text{ if } x < 0$$

Because this requires two different processes or pieces, the absolute value function is an example of a piecewise function. A **piecewise function** is a function in which more than one formula is used to define the output over different pieces of the domain.

We use piecewise functions to describe situations in which a rule or relationship changes as the input value crosses certain “boundaries.” For example, we often encounter situations in business for which the cost per piece of a certain item is discounted once the number ordered exceeds a certain value. Tax brackets are another real-world example of piecewise functions. For example, consider a simple tax system in which incomes up to $10,000 are taxed at 10%, and any additional income is taxed at 20%. The tax on a total income S would be $0.1S$ if $S \leq \$10{,}000$ and $\$1000 + 0.2(S - \$10{,}000)$ if $S > \$10{,}000$.

Note:
A piecewise function is a function in which more than one formula is used to define the output. Each formula has its own domain, and the domain of the function is the union of all these smaller domains. We notate this idea like this:
Equation:

$$f(x) = \begin{array}{ll} \text{formula 1} & \text{if } x \text{ is in domain 1} \\ \text{formula 2} & \text{if } x \text{ is in domain 2} \\ \text{formula 3} & \text{if } x \text{ is in domain 3} \end{array}$$

In piecewise notation, the absolute value function is
Equation:

$$|x| = \begin{cases} x & \text{if } \ x \geq 0 \\ -x & \text{if } \ x < 0 \end{cases}$$

Note:
Given a piecewise function, write the formula and identify the domain for each interval.

1. Identify the intervals for which different rules apply.
2. Determine formulas that describe how to calculate an output from an input in each interval.
3. Use braces and if-statements to write the function.

Example:
Writing Piecewise Functions
Exercise:

Problem:
Writing a Piecewise Function

A museum charges $5 per person for a guided tour with a group of 1 to 9 people or a fixed $50 fee for a group of 10 or more people. Write a function relating the number of people, $n,$ to the cost, $C.$

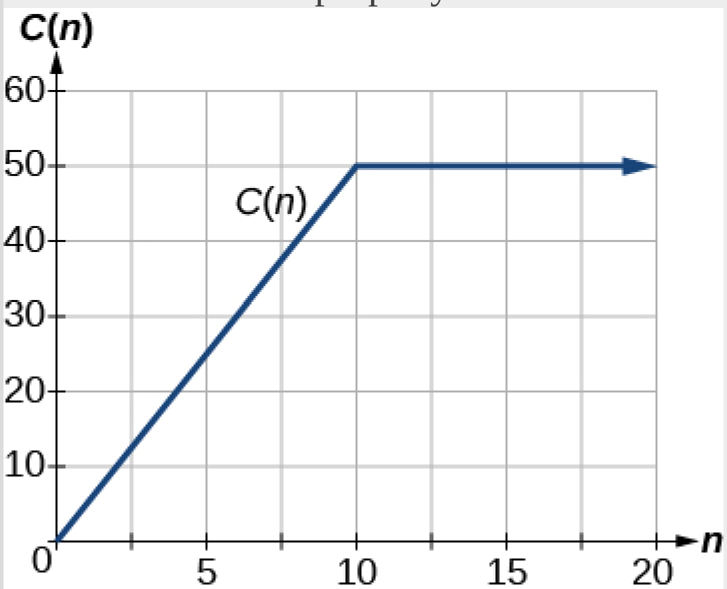
Solution:

Two different formulas will be needed. For n-values under 10, $C = 5n.$ For values of n that are 10 or greater, $C = 50.$
Equation:

$$C(n) = \begin{cases} 5n & \text{if} \quad 0 < n < 10 \\ 50 & \text{if} \quad\quad n \geq 10 \end{cases}$$

Analysis

The function is represented in [link]. The graph is a diagonal line from $n = 0$ to $n = 10$ and a constant after that. In this example, the two formulas agree at the meeting point where $n = 10,$ but not all piecewise functions have this property.



Example:
Exercise:

Problem:
Working with a Piecewise Function

A cell phone company uses the function below to determine the cost, $C,$ in dollars for g gigabytes of data transfer.

Equation:

$$C(g) = \begin{cases} 25 & \text{if} \quad 0 < g < 2 \\ 25 + 10(g - 2) & \text{if} \quad g \geq 2 \end{cases}$$

Find the cost of using 1.5 gigabytes of data and the cost of using 4 gigabytes of data.

Solution:

To find the cost of using 1.5 gigabytes of data, $C(1.5),$ we first look to see which part of the domain our input falls in. Because 1.5 is less than 2, we use the first formula.

Equation:

$$C(1.5) = \$25$$

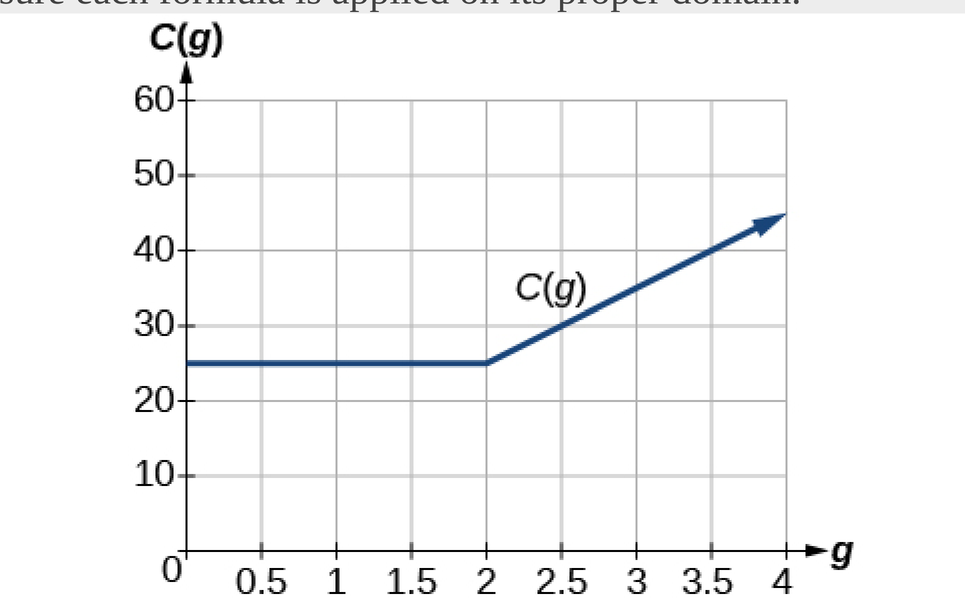
To find the cost of using 4 gigabytes of data, $C(4),$ we see that our input of 4 is greater than 2, so we use the second formula.

Equation:

$$C(4) = 25 + 10(4 - 2) = \$45$$

Analysis

The function is represented in [link]. We can see where the function changes from a constant to a shifted and stretched identity at $g = 2$. We plot the graphs for the different formulas on a common set of axes, making sure each formula is applied on its proper domain.



Note:
Given a piecewise function, sketch a graph.

1. Indicate on the *x*-axis the boundaries defined by the intervals on each piece of the domain.
2. For each piece of the domain, graph on that interval using the corresponding equation pertaining to that piece. Do not graph two functions over one interval because it would violate the criteria of a function.

Example:
Exercise:

Problem:
Graphing a Piecewise Function

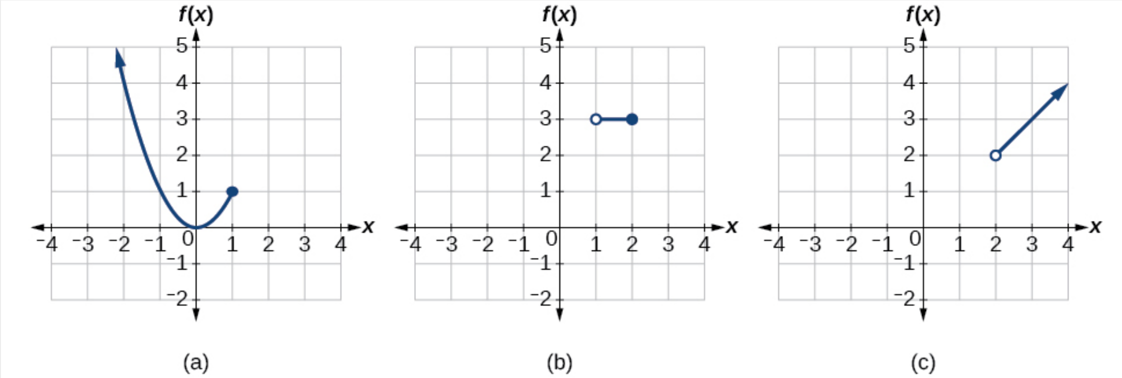
Sketch a graph of the function.
Equation:

$$f(x)=\begin{array}{ccc} x^2 & \text{if} & x\le 1 \\ 3 & \text{if} & 1<x\le 2 \\ x & \text{if} & x>2 \end{array}$$

Solution:

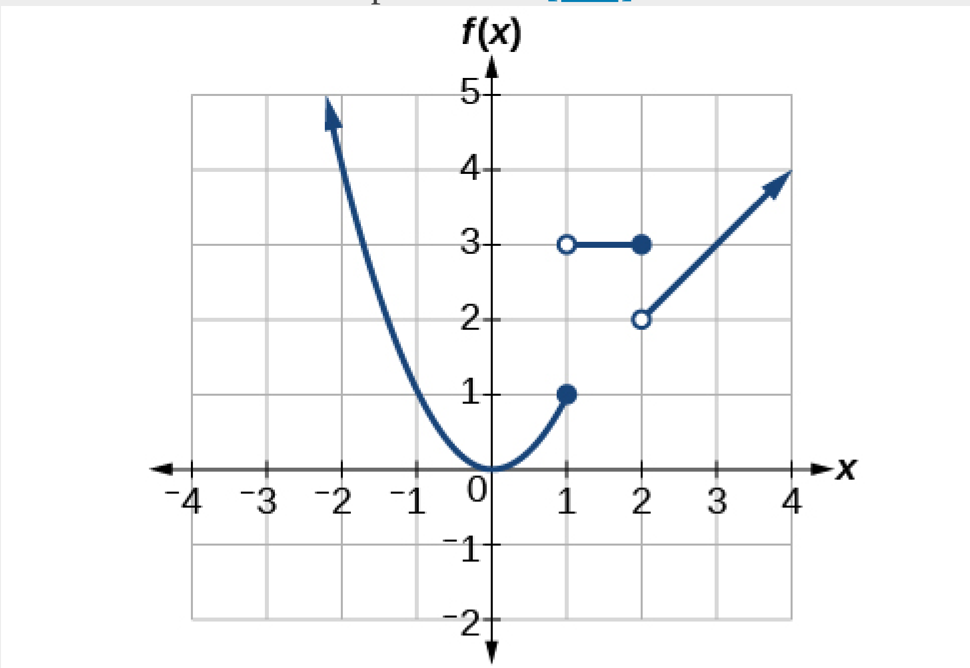
Each of the component functions is from our library of toolkit functions, so we know their shapes. We can imagine graphing each function and then limiting the graph to the indicated domain. At the endpoints of the domain, we draw open circles to indicate where the endpoint is not included because of a less-than or greater-than inequality; we draw a closed circle where the endpoint is included because of a less-than-or-equal-to or greater-than-or-equal-to inequality.

[link] shows the three components of the piecewise function graphed on separate coordinate systems.



(a) $f\left(x\right)={x}^{2}$ if $x\le 1$; (b) $f\left(x\right)=3$ if $1<x\le 2$; (c) $f\left(x\right)=x$ if $x>2$

Now that we have sketched each piece individually, we combine them in the same coordinate plane. See [link].



Analysis

Note that the graph does pass the vertical line test even at $x=1$ and $x=2$ because the points $\left(1,3\right)$ and $\left(2,2\right)$ are not part of the graph of the

function, though $(1, 1)$ and $(2, 3)$ are.

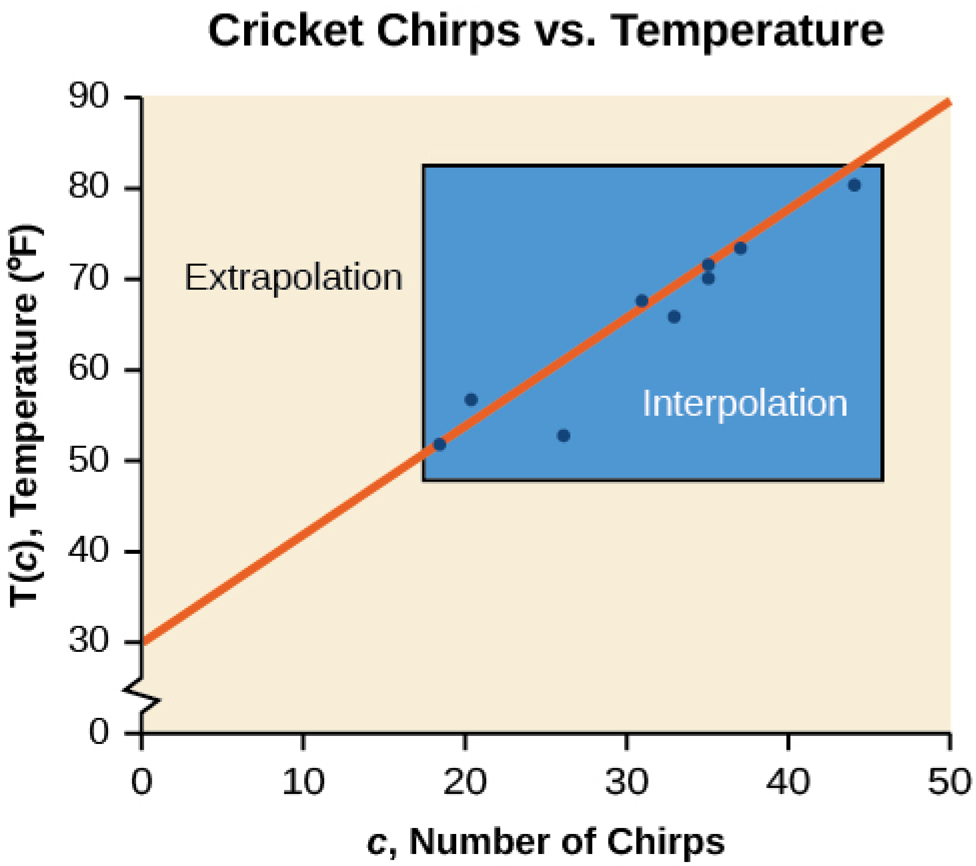
Recognizing Interpolation or Extrapolation [footnote]

Section material derived from Openstax Precalculus: Fitting Linear Models to Data-Fitting Linear Models to Data

While the data for most examples does not fall perfectly on the line, the equation is our best guess as to how the relationship will behave outside of the values for which we have data. We use a process known as **interpolation** when we predict a value inside the domain and range of the data. The process of **extrapolation** is used when we predict a value outside the domain and range of the data.

[link] compares the two processes for the cricket-chirp data addressed in [link]. We can see that interpolation would occur if we used our model to predict temperature when the values for chirps are between 18.5 and 44. Extrapolation would occur if we used our model to predict temperature when the values for chirps are less than 18.5 or greater than 44.

There is a difference between making predictions inside the domain and range of values for which we have data and outside that domain and range. Predicting a value outside of the domain and range has its limitations. When our model no longer applies after a certain point, it is sometimes called **model breakdown**. For example, predicting a cost function for a period of two years may involve examining the data where the input is the time in years and the output is the cost. But if we try to extrapolate a cost when $x = 50$, that is in 50 years, the model would not apply because we could not account for factors fifty years in the future.



Interpolation occurs within the domain and range of the provided data whereas extrapolation occurs outside.

Note:
Different methods of making predictions are used to analyze data.

The method of **interpolation** involves predicting a value inside the domain and/or range of the data.
The method of **extrapolation** involves predicting a value outside the domain and/or range of the data.
Model breakdown occurs at the point when the model no longer applies.

Example:
Understanding Interpolation and Extrapolation
Exercise:

Problem:

Use the cricket data from [link] to answer the following questions:

a. Would predicting the temperature when crickets are chirping 30 times in 15 seconds be interpolation or extrapolation? Make the prediction, and discuss whether it is reasonable.
b. Would predicting the number of chirps crickets will make at 40 degrees be interpolation or extrapolation? Make the prediction, and discuss whether it is reasonable.

Solution:

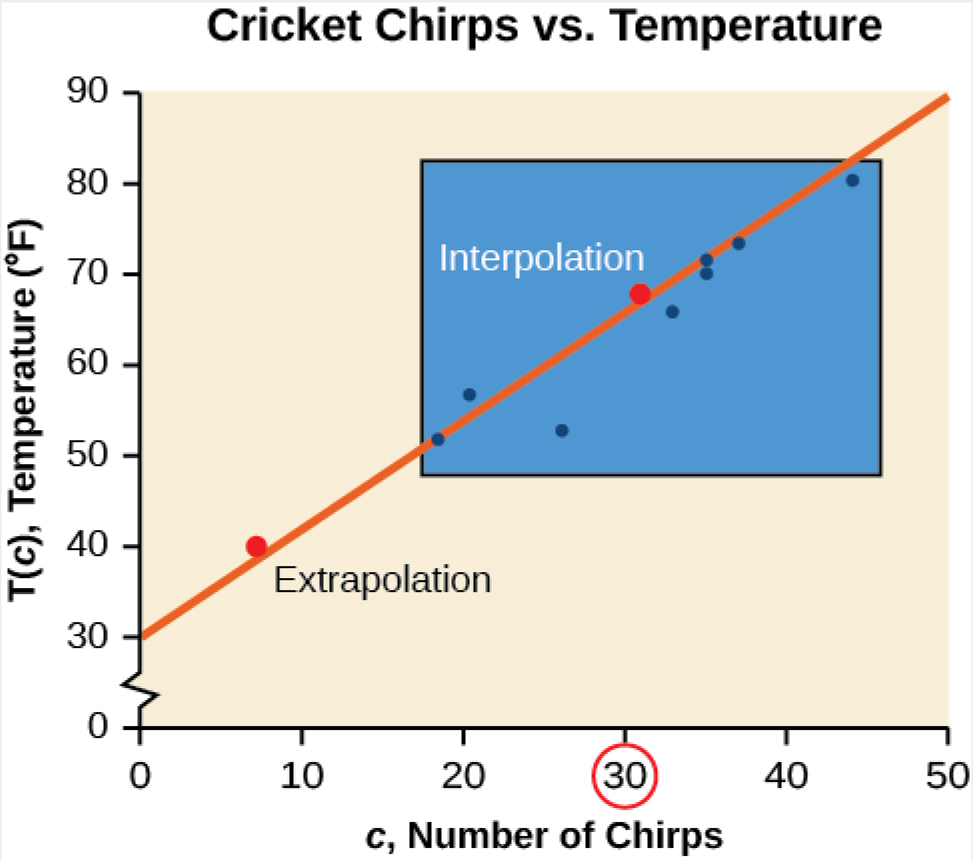
a. The number of chirps in the data provided varied from 18.5 to 44. A prediction at 30 chirps per 15 seconds is inside the domain of our data, so would be interpolation. Using our model:
Equation:

$$\begin{aligned} T(30) &= 30 + 1.2(30) \\ &= 66 \text{ degrees} \end{aligned}$$

Based on the data we have, this value seems reasonable.
b. The temperature values varied from 52 to 80.5. Predicting the number of chirps at 40 degrees is extrapolation because 40 is outside the range of our data. Using our model:
Equation:

$$\begin{aligned} 40 &= 30 + 1.2c \\ 10 &= 1.2c \\ c &\approx 8.33 \end{aligned}$$

We can compare the regions of interpolation and extrapolation using [link].



Analysis

Our model predicts the crickets would chirp 8.33 times in 15 seconds. While this might be possible, we have no reason to believe our model is valid outside the domain and range. In fact, generally crickets stop chirping altogether below around 50 degrees.

Key Concepts

- **Simple interest is represented and calculated with the equation**

$$I = P \cdot r \cdot t$$

- **Compound interest is represented and calculated with the equation**

$$B = P(r + 1)^t$$

Glossary

Compound Interest
: A method of charging interest on a loan or interest earned in a bank account which uses the previous year's balance, the rate of interest, and the time spent in the account as the information needed to calculate the interest made. This relationship can be modeled with a non-linear, or exponential, equation.

extrapolation
: predicting a value outside the domain and range of the data

interpolation
: predicting a value inside the domain and range of the data

Intercepts of a Line
: Each of the points at which a line crosses the *x*-axis and the *y*-axis is called an intercept of the line.

Piecewise function
: a function in which more than one formula is used to define the output

Simple Interest

A method of charging interest on a loan or interest earned in a bank account which uses the initial deposit, the rate of interest, and the time spent in the account as the information needed to calculate the interest made.

Proportions: Lesson 9.B

Note:
In case you missed something in class, or just want to review a specific topic covered in this Module, here is a list of topics covered:

Solving Proportions [footnote]

Section material derived from Openstax Elementary Algebra: Rational Expressions and Equations-Solve Proportion and Similar Figure Applications

When two rational expressions are equal, the equation relating them is called a *proportion*.

Note:
A **proportion** is an equation of the form $\frac{a}{b} = \frac{c}{d}$, where $b \neq 0, d \neq 0.$
The proportion is read "a is to b, as c is to d."

The equation $\frac{1}{2} = \frac{4}{8}$ is a proportion because the two fractions are equal.
The proportion $\frac{1}{2} = \frac{4}{8}$ is read "1 is to 2 as 4 is to 8."

Example:
Solving Proportions
Exercise:

Problem: Solve the proportion: $\frac{x}{63} = \frac{4}{7}$.

Solution:
Solution

		$\frac{x}{63} = \frac{4}{7}$
To isolate x, multiply both sides by the LCD, 63.		$63\left(\frac{x}{63}\right) = 63\left(\frac{4}{7}\right)$
Simplify.		$x = \frac{9 \cdot \cancel{7} \cdot 4}{\cancel{7}}$
Divide the common factors.		$x = 36$
Check. To check our answer, we substitute into the original proportion.		
	$\frac{x}{63} = \frac{4}{7}$	
Substitute $x = 36$.	$\frac{36}{63} \stackrel{?}{=} \frac{4}{7}$	

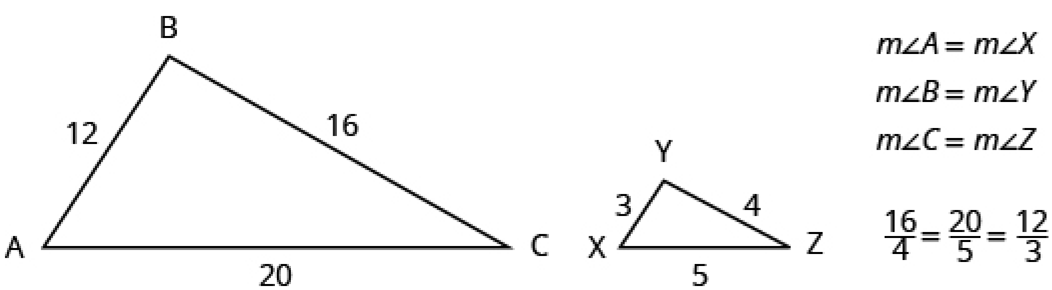
Show common factors.	$\frac{4 \cdot 9}{7 \cdot 9} \stackrel{?}{=} \frac{4}{7}$	
Simplify.	$\frac{4}{7} = \frac{4}{7} \checkmark$	

Similar Triangles [footnote]

Section material derived from Openstax Prealgebra: Math Models and Geometry-Use Properties of Angles Triangles and the Pythagorean Theorem

When we use a map to plan a trip, a sketch to build a bookcase, or a pattern to sew a dress, we are working with similar figures. In geometry, if two figures have exactly the same shape but different sizes, we say they are **similar figures**. One is a scale model of the other. The corresponding sides of the two figures have the same ratio, and all their corresponding angles are have the same measures.

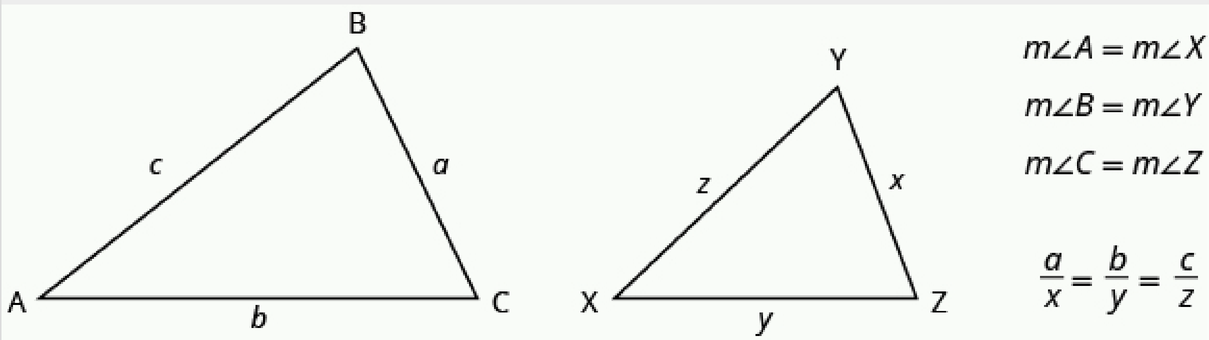
The two triangles in [link] are similar. Each side of ΔABC is four times the length of the corresponding side of ΔXYZ and their corresponding angles have equal measures.



ΔABC and ΔXYZ are similar triangles. Their corresponding sides have the same ratio and the

corresponding angles have the same measure.

Note:
If two triangles are similar, then their corresponding angle measures are equal and their corresponding side lengths are in the same ratio.



The length of a side of a triangle may be referred to by its endpoints, two vertices of the triangle. For example, in ΔABC:

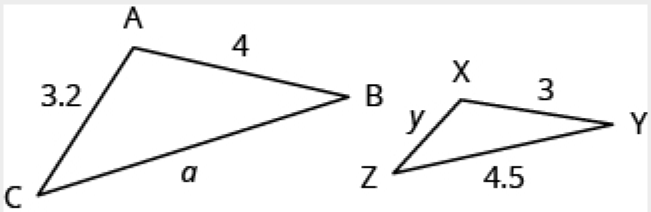
$$\text{the length } a \text{ can also be written } BC$$
$$\text{the length } b \text{ can also be written } AC$$
$$\text{the length } c \text{ can also be written } AB$$

We will often use this notation when we solve similar triangles because it will help us match up the corresponding side lengths.

Example:
Solving Similar Triangles
Exercise:

Problem:

ΔABC and ΔXYZ are similar triangles. The lengths of two sides of each triangle are shown. Find the lengths of the third side of each triangle.



Solution:
Solution

Step 1. **Read** the problem. Draw the figure and label it with the given information.	The figure is provided.
Step 2. **Identify** what you are looking for.	The length of the sides of similar triangles
Step 3. **Name.** Choose a variable to represent it.	Let *a* = length of the third side of ΔABC *y* = length of the third side ΔXYZ
Step 4. **Translate.**	
The triangles are similar, so the corresponding sides are in the same ratio. So	

Equation:

$$\frac{AB}{XY} = \frac{BC}{YZ} = \frac{AC}{XZ}$$

Since the side $AB = 4$ corresponds to the side $XY = 3$, we will use the ratio $\frac{AB}{XY} = \frac{4}{3}$ to find the other sides.

Be careful to match up corresponding sides correctly.

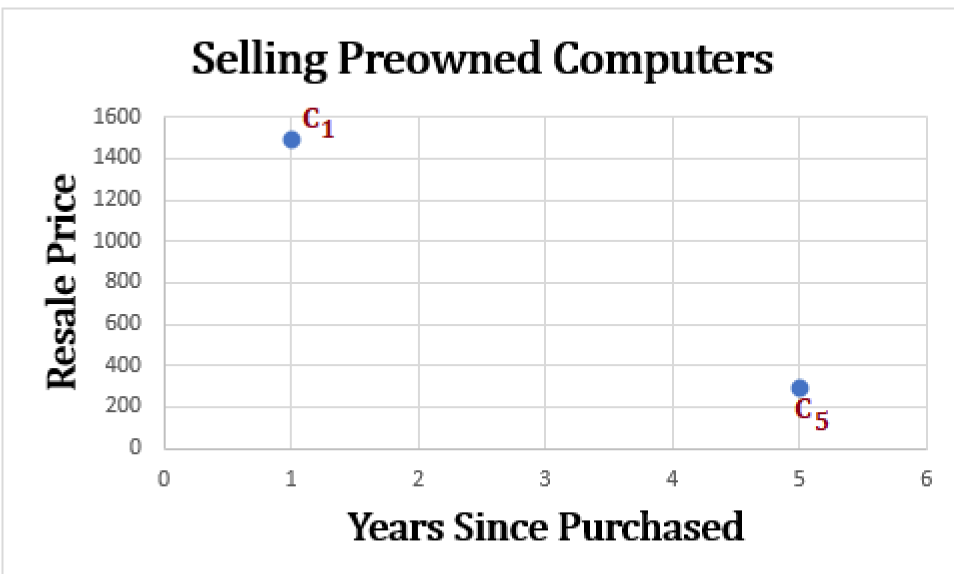
	To find a:	To find y:
sides of large triangle →	$\frac{AB}{XY} = \frac{BC}{YZ}$	$\frac{AB}{XY} = \frac{AC}{XZ}$
sides of small triangle →	$\frac{4}{3} = \frac{a}{4.5}$	$\frac{4}{3} = \frac{3.2}{y}$

Step 5. **Solve** the equation.	$3a = 4(4.5)$ $3a = 18$ $a = 6$ $4y = 3(3.2)$ $4y = 9.6$ $y = 2.4$
Step 6. **Check:** $\frac{4}{3} \stackrel{?}{=} \frac{6}{4.5}$ $4(4.5) \stackrel{?}{=} 6(3)$ $18 = 18 \checkmark$ $\frac{4}{3} \stackrel{?}{=} \frac{3.2}{2.4}$ $4(2.4) \stackrel{?}{=} 3.2(3)$ $9.6 = 9.6 \checkmark$	
Step 7. **Answer** the question.	The third side of ΔABC is 6 and the third side of ΔXYZ is 2.4.

Applications of Similar Triangles [footnote]

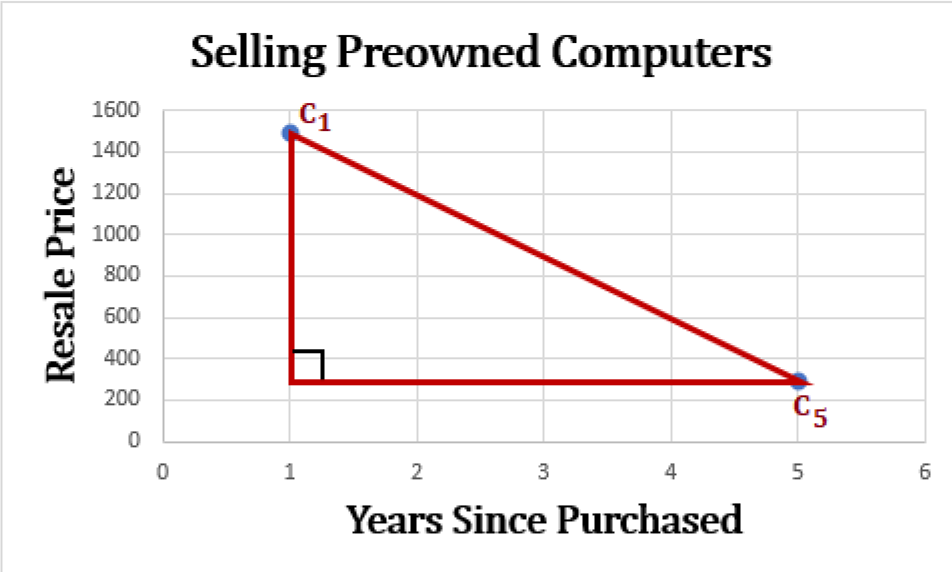
This material was created by Amanda Towry .

Two neighbors in your apartment building sold their old computers last week. Both were in very good condition and with similar specifications and upgrades. One of the computers was 1 year old and sold for $1,500 and the other was 5 years old and sold for $300. This information can be graphed in order to look at how the value of computers changes over time. You have a computer that is currently 3 years old and are thinking about selling it so you can get a new one. Can you use the information you have about your neighbors selling prices to predict what your computer is currently worth selling for? Of course!



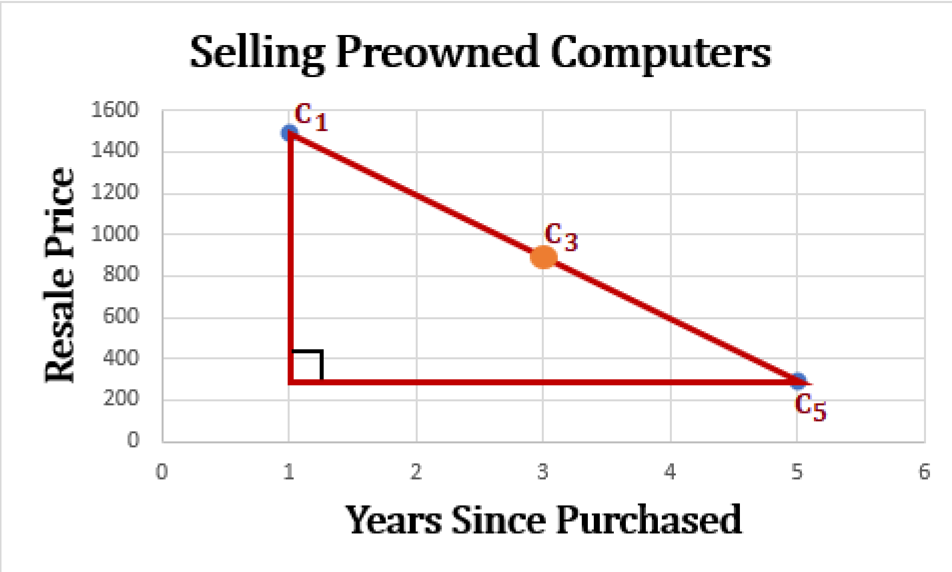
Where C1= the value of the computer after 1 year of ownership and C5 is the value of the computer after 5 years of ownership.

We can create a right triangle with the two data points we have



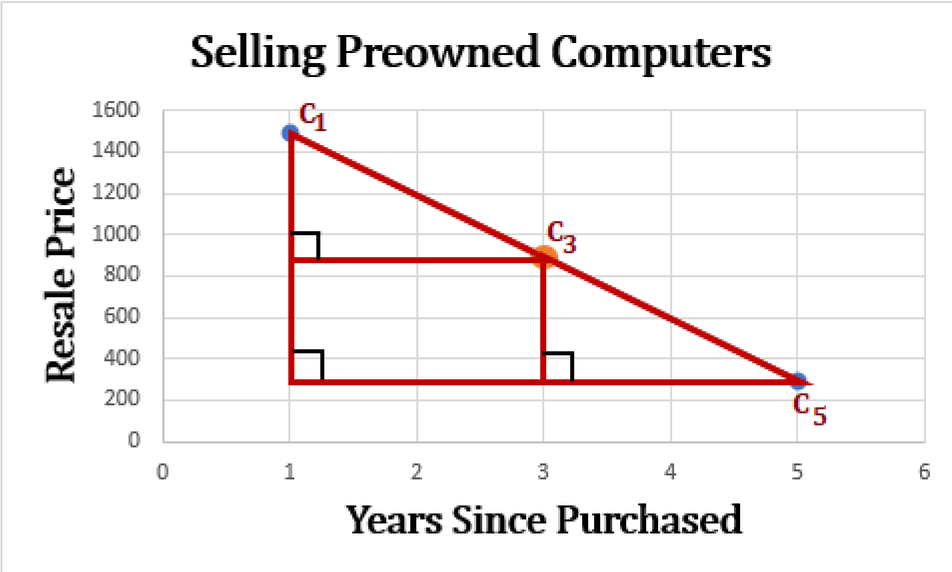
Where C1 is the value of the computer after 1 year of ownership, C3 is the value of the computer after 3 years of ownership, and C5 is the value of the computer after 5 years of ownership.

Knowing that your computer is 3 years old and having a general trend of depreciation for the computers, you can add your data to the table and try to predict the resale value of it using the power of similar triangles.



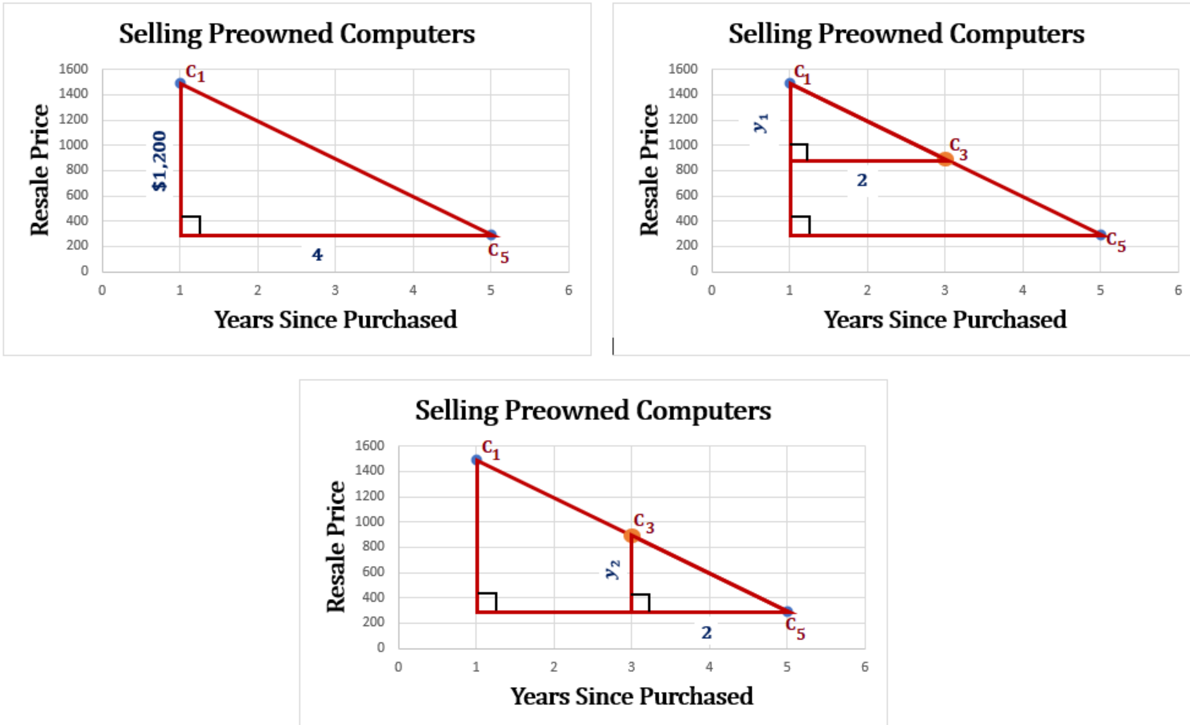
Where C1 is the value of the computer after 1 year of ownership, C3 is the value of the computer after 3 years of ownership, and C5 is the value of the computer after 5 years of ownership.

We can create two smaller right triangles that each connect between a data point and the placement of your computer at 3 years old on the trend line.



Where C1 is the value of the computer after 1 year of ownership, C3 is the value of the computer after 3 years of ownership, and C5 is the value of the computer after 5 years of ownership.

"How does this help?!" you might ask. Well, since we have created similar triangles, we can set up proportions between two of the similar triangles to determine the possible sale price of your computer. How? Well, first you need to label all of the sides of the triangles that we could possibly use in proportions. Keep in mind that in this case, the hypotenuses do not give any useful information for us to use, so we will not label them with values at all.



[link] the large triangle connects from the computer of year 1 to the computer of year 5. There is a difference in selling price (the y-value) of $1,200 and a difference in years (the x-value) of 4 years between C_1 and C_5. In the top right of [link],the difference in years owned is 2 and we don't know the difference in selling price betweenC_1 and C_3; that's what we are looking for(y_1)! In the bottom center, the difference in years owned is 2 years and the difference in selling prices from C_3 to $C_5$is, again, unknown (y_2).

Using what we know about similar triangles, we can find the price difference for both the top and bottom triangle.

Equation:

$$\frac{2}{4} = \frac{y_1}{\$1{,}2000}$$

Then cross multiplication will give us a single variable equation to solve.

Equation:

$$(2\ )(\$\ 1{,}200\ ){=}(\ 4\ )(\ 2{,}400)$$

Equation:

$$\$2{,}400 = 4y_1$$

Equation:

$$\frac{\$2{,}400}{4} = y_1$$

Equation:

$$y_1 = \$600$$

Is the selling price for a 3 year old computer $\$600$? Not exactly. Remember that the 1 year old computer was sold for $\$1{,}500$, not $\$1{,}200$. In order to find the probable selling price of the computer, you must subtract the value found for y_1, from the original price of $\$1{,}500$.

Equation:

$$\$1{,}500 - \$600 = \$\ 900$$

Another possible route to the answer was tp solve for the propportion including the bottom triangle. You would ge tthe same answer, but here is the set up:

Equation:

$$\frac{2}{4} = \frac{y_2}{\$1{,}2000}$$

Once this equation is solved for, the result can be **added** to the lower sale price at year 5 (C_5) .

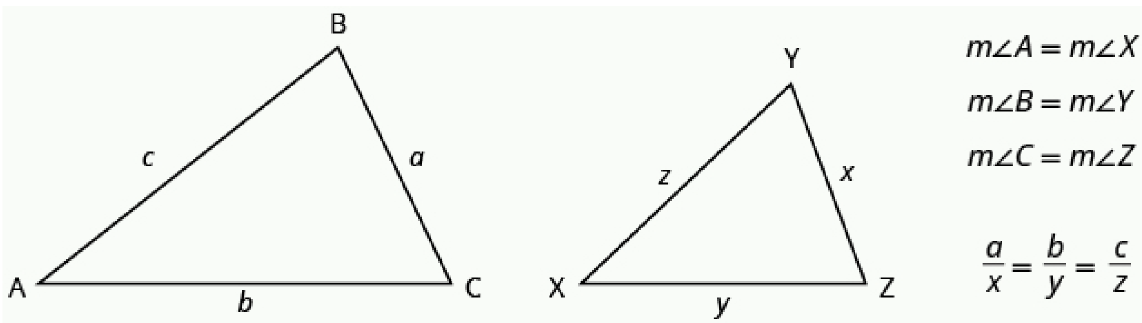
Equation:

$$\$300 + \$600 = \$900$$

Thus, a reasonable price to sell your used computer at is $900.

Key Concepts

- **A proportion is an equation of the form** $\frac{a}{b} = \frac{c}{d}$, where $b \neq 0, d \neq 0$.
- **If two triangles are similar, then their corresponding angle measures are equal and their corresponding side lengths are in the same ratio.**



Glossary

Proportion
: A proportion is an equation of the form $\frac{a}{b} = \frac{c}{d}$, where $b \neq 0$, $d \neq 0$ .The proportion states two ratios or rates are equal. The proportion is read "a is to b, as c is to d

Similar Figures
: Two figures are similar if the measures of their corresponding angles are equal and their corresponding sides are in the same ratio.

Scatterplot Data and its Trends: Lessons 9.C - 9.D
In this section, you will:

- Draw and interpret scatter plots.
- Find the line of best fit.
- Distinguish between linear and nonlinear relations.
- Use a linear model to make predictions.

This correlates to Lessons 9A-D of Corequisite MAT 1043(QR) and NCBO.

Note:
In case you missed something in class, or just want to review a specific topic covered in this Module, here is a list of topics covered:

1. Univariate Vs. Biavariate Data [link]
2. Drawing and Interpreting Scatterplots [link]
3. Finding the Line of Best Fit [link]
4. Distinguishing between Linear and Non-Linear Models [link]
5. Predicting with a Regression Line [link]
6. Key Concepts [link]

Univariate Vs. Bivariate Data [footnote]

This material was created by Amanda Towry .

Univariate Data
Univariate means "one variable".

- Involves a single variable
- Does not deal with causes or relationships.
- Allows for the description of data collected.
- Is quantitative in nature
- Uses measures of central tendencies such as the mean, median, mode, quartiles, and the standard deviation
- Results can be shown in Bar graphs, Histograms, Pie charts, Line graphs, and Box plots.

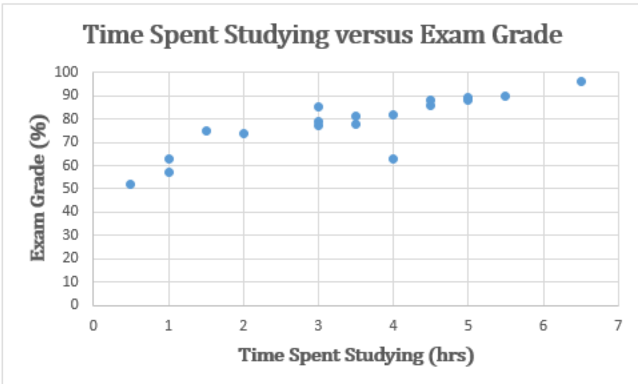
Bivariate Data
Bivariate means "two variables".

- Involves two variables.
- Deals with causes or relationships.
- Allows for explanation of data
- Uses analysis of two variables separately but simultaneously
- Uses correlation between two variables
- Develops comparisons, relationships, causes, and explanations.
- Results can be shown in tables where one variable is dependent on the values of another variable

What is a Scatter Plot?

A scatter plot is a data graph that uses points to represent the relationship between two variables of interest. Although the horizontal axis is still associated with the independent variable and the vertical axis with the dependent variable, they do not have the same assumed relationship we normally think of with more common data graphs. Instead, by using a scatter plot, we are attempting to **infer** a relationship between the two variables.

The figure below is an example of a scatter plot modeling how time spent studying compares to exam scores.



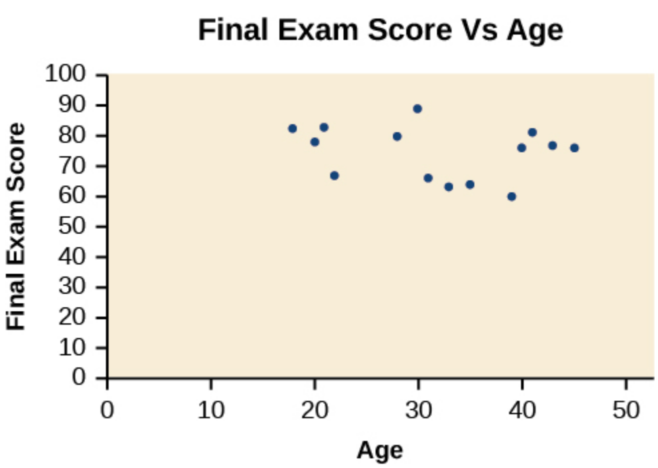
Many people would want to immediately say that more time spent studying **causes** higher exam scores. However, when we are using scatterplots in the coming sections, it is important that they should never be read as a definite relationship. Instead of a **cause** we must emphasize a **correlation** instead.

Correlation is a description and measure of how well the change in two separate variables match with each other. Correlation between two variables does not mean that one necessarily causes the other. Thus, in the math world, we emphasize that **"Correlation does not equal causation."**

Drawing and Interpreting Scatter Plots [footnote]

Section material derived from Openstax Precalculus: Linear Functions-Fitting Linear Models to Data

A scatter plot is a graph of plotted points that may show a relationship between two sets of data. If the relationship is from a linear model, or a model that is nearly linear, the professor can draw conclusions using his knowledge of linear functions. [link] shows a sample scatter plot.



A scatter plot of age and final exam score variables

Notice this scatter plot does *not* indicate a linear relationship. The points do not appear to follow a trend. In other words, there does not appear to be a relationship between the age of the student and the score on the final exam.

Example:
Using a Scatter Plot to Investigate Cricket Chirps
Exercise:

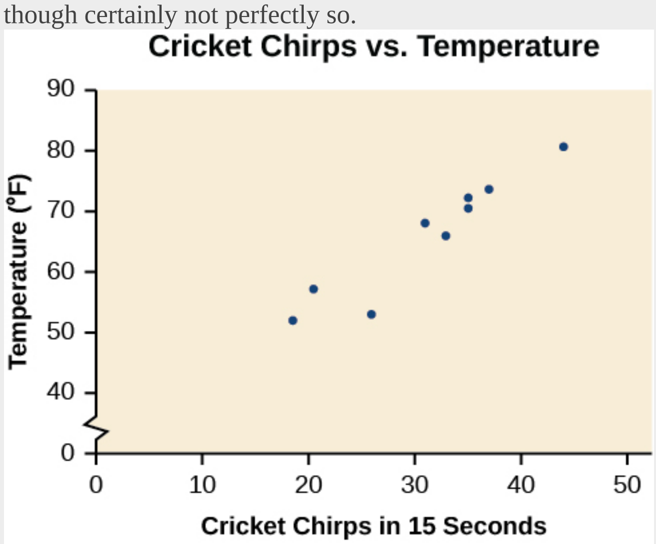
Problem:

[link] shows the number of cricket chirps in 15 seconds, for several different air temperatures, in degrees Fahrenheit[footnote]. Plot this data, and determine whether the data appears to be linearly related.
Selected data from http://classic.globe.gov/fsl/scientistsblog/2007/10/. Retrieved Aug 3, 2010

Chirps	44	35	20.4	33	31	35	18.5	37	26
Temperature	80.5	70.5	57	66	68	72	52	73.5	53

Solution:

Plotting this data, as depicted in [link] suggests that there may be a trend. We can see from the trend in the data that the number of chirps increases as the temperature increases. The trend appears to be roughly linear, though certainly not perfectly so.

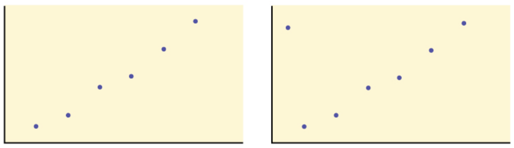


A scatter plot shows the **direction** of a relationship between the variables. A clear direction happens when there is either:

- High values of one variable occurring with high values of the other variable or low values of one variable occurring with low values of the other variable.
- High values of one variable occurring with low values of the other variable.

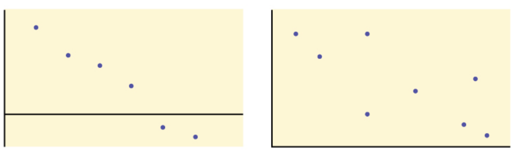
You can determine the **strength** of the relationship by looking at the scatter plot and seeing how close the points are to a line, a power function, an exponential function, or to some other type of function. For a linear relationship there is an exception. Consider a scatter plot where all the points fall on a horizontal line providing a "perfect fit." The horizontal line would in fact show no relationship.

When you look at a scatterplot, you want to notice the **overall pattern** and any **deviations** from the pattern. The following scatterplot examples illustrate these concepts.



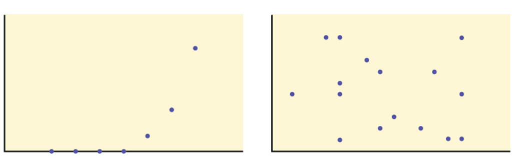
(a) Positive linear pattern (strong) (b) Linear pattern w/ one deviation

The points lie close to a straight line, which has a positive slope. This shows that as one variable **increases** (the independent variable), the other **increases** as well (dependent variable).



(a) Negative linear pattern (strong) (b) Negative linear pattern (weak)

(a) The points lie close to a straight line, which has a negative slope. This shows that as one variable **increases** (the independent variable), the other **decreases** (the dependent variable).



(a) Exponential growth pattern (b) No pattern

When a scatter plot is constructed, the behavior of the data can be said to have a correlation.

Correlation is a way to define the relationship between the variables being compared.

Finding the Line of Best Fit [footnote]

Section material derived from Openstax Precalculus: Linear Functions-Fitting Linear Models to Data

Once we recognize a need for a linear function to model that data, the natural follow-up question is "what is that linear function?" One way to approximate our linear function is to sketch the line that seems to best fit the data. Then we can extend the line until we can verify the y-intercept. We can approximate the slope of the line by extending it until we can estimate the $\frac{\text{rise}}{\text{run}}$.

Example:
Finding a Line of Best Fit
Exercise:

Problem: Find a linear function that fits the data in [link] by "eyeballing" a line that seems to fit.

Solution:

On a graph, we could try sketching a line.

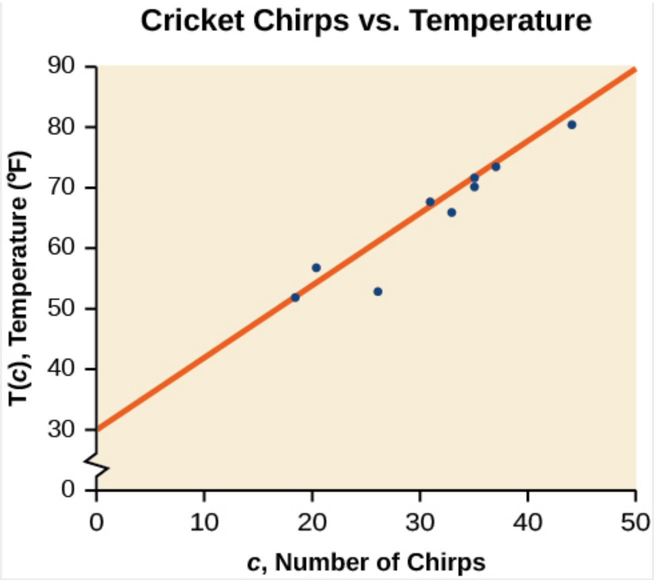
Using the starting and ending points of our hand drawn line, points (0, 30) and (50, 90), this graph has a slope of
Equation:

$$m = \frac{60}{50} = 1.2$$

and a y-intercept at 30. This gives an equation of
Equation:

$$T(c) = 1.2c + 30$$

where c is the number of chirps in 15 seconds, and $T\left(c\right)$ is the temperature in degrees Fahrenheit. The resulting equation is represented in [link].



Analysis

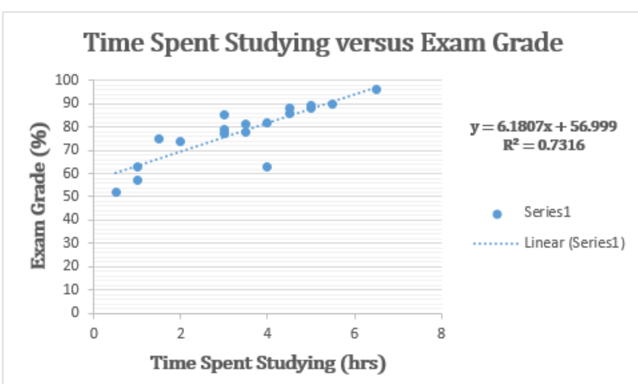
This linear equation can then be used to approximate answers to various questions we might ask about the trend.

Interpreting the Line of Best Fit[footnote]

Section material created by Amanda Towry

When a line of best fit is applied to a scatterplot, there are a couple of mathematical values that give quite a bit of information about the data relationship and what it means.

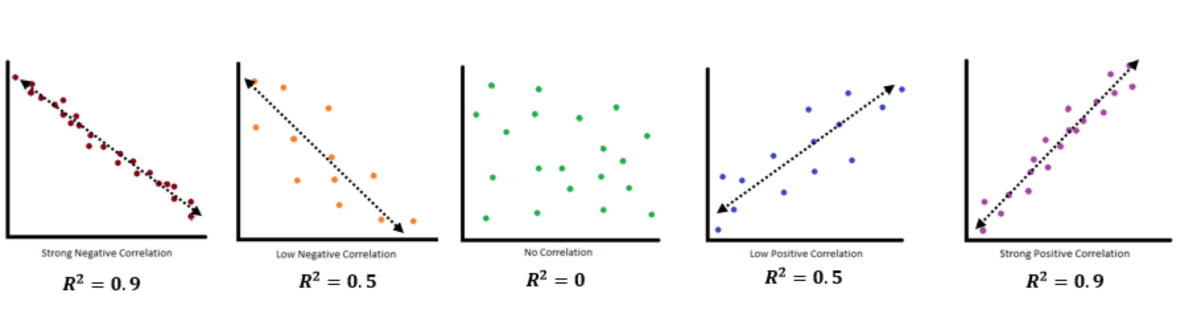
Coefficient of Determination
When a trend line is first calculated by Excel for a scatter plot, it can apply an equation to the graph. The equation defines the line created for the data behavior and is accompanied by another value of R^2



Notice in the upper right there is an equation with a value below of r squared. This is the coefficient of determination.

Coefficient of Determination is a value R^2, between 0 and 1, that gives a numerical description of how well the regression model fits or predicts the data it was derived from.

- When R^2 is close to 0: the regression model does not fit the data at all.
- When R^2 is close to 1: the regression model fits the data exactly.
- Is the square of the correkation coefficient.
- Is **always** a positive value.

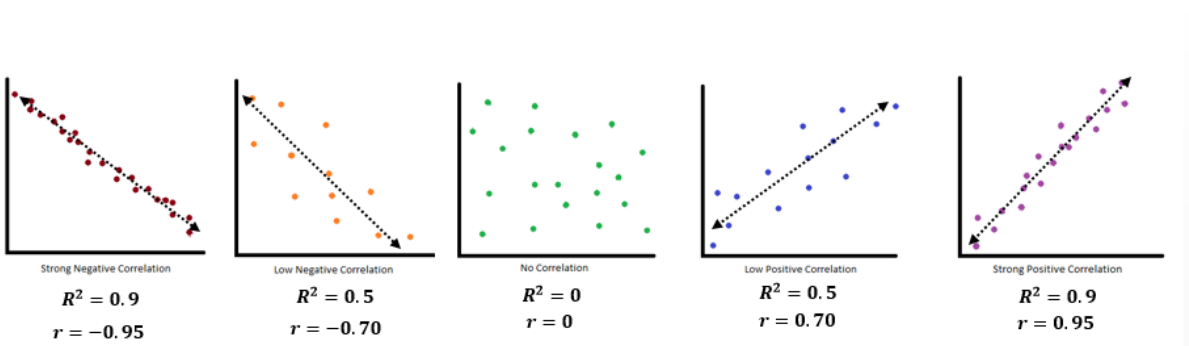


Correlation Coefficient
The correlation coefficient isn't given to you automatically when you are analyzing linear data with Excel. You must calculate it from the data you **are** given. Taking the square root of the coefficient of determination will give you the magnitude of the correlation coefficient. However, you must pay attention to the trend of the data to determine whether it is a negative or positive value. The slope of the line gives you a hint.
Equation:

$$\mathrm{r} = \sqrt{R^2}$$

The **correlation coefficient** is a value,r, between –1 and 1.

- $r > 0$ suggests a positive (increasing) relationship
- $r < 0$ suggests a negative (decreasing) relationship
- The closer the value is to 0, the more scattered the data.
- The closer the value is to 1 or –1, the less scattered the data is.

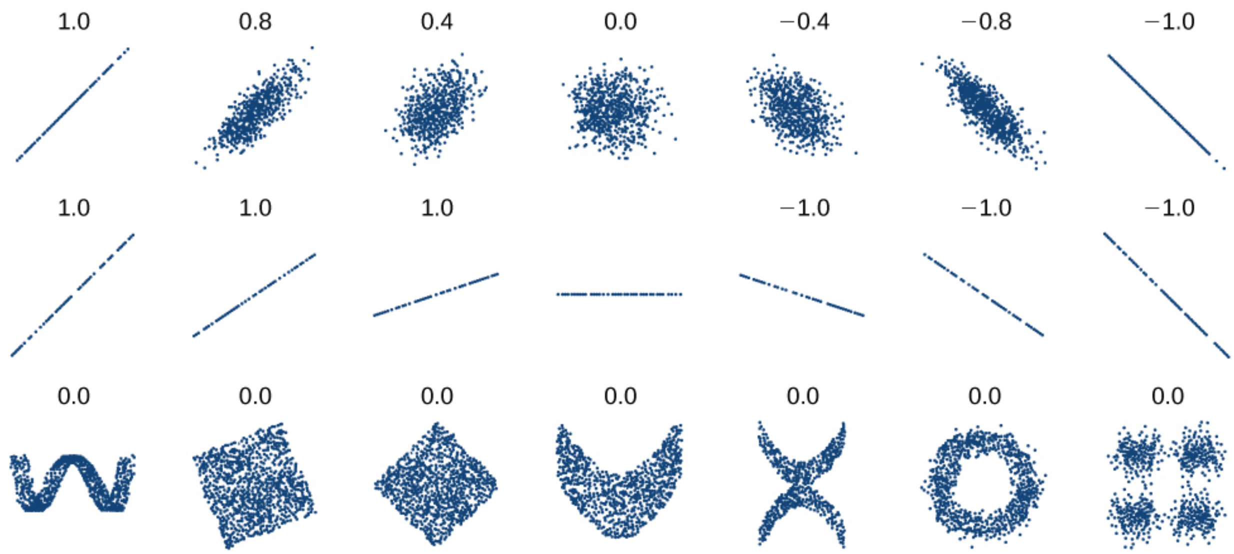


Distinguishing Between Linear and Non-Linear Models [footnote]

Section material derived from Openstx Precalculus: Linear Functions-Fitting Linear Models to Data

As we saw above with the cricket-chirp model, some data exhibit strong linear trends, but other data, like the final exam scores plotted by age, are clearly nonlinear. Most calculators and computer software can also provide us with the **correlation coefficient**, which is a measure of how closely the line fits the data. Many graphing calculators require the user to turn a ”diagnostic on” selection to find the correlation coefficient, which mathematicians label as r. The correlation coefficient provides an easy way to get an idea of how close to a line the data falls.

We should compute the correlation coefficient only for data that follows a linear pattern or to determine the degree to which a data set is linear. If the data exhibits a nonlinear pattern, the correlation coefficient for a linear regression is meaningless. To get a sense for the relationship between the value of r and the graph of the data, [link] shows some large data sets with their correlation coefficients. Remember, for all plots, the horizontal axis shows the input and the vertical axis shows the output.



Plotted data and related correlation coefficients. (credit: “DenisBoigelot,” Wikimedia Commons)

Predicting with a Regression Line [footnote]

Section material derived from Openstax Precalculus: Linear Functions-Fitting Linear Models to Data

Once we determine that a set of data is linear using the correlation coefficient, we can use the regression line to make predictions. As we learned above, a regression line is a line that is closest to the data in the scatter plot, which means that only one such line is a best fit for the data.

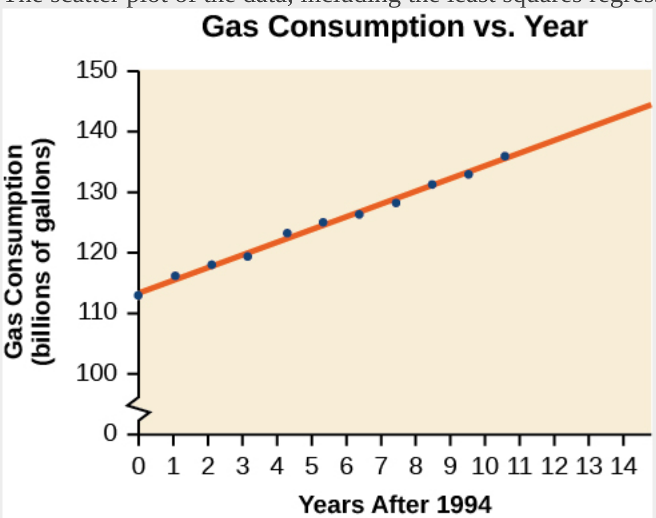
Example:
Using a Regression Line to Make Predictions
Exercise:

Problem:

Gasoline consumption in the United States has been steadily increasing. Consumption data from 1994 to 2004 is shown in [link][footnote]. Determine whether the trend is linear, and if so, find a model for the data. Use the model to predict the consumption in 2008.
http://www.bts.gov/publications/national_transportation_statistics/2005/html/table_04_10.html

Year	'94	'95	'96	'97	'98	'99	'00	'01	'02	'03
Consumption (billions of gallons)	113	116	118	119	123	125	126	128	131	133

The scatter plot of the data, including the least squares regression line, is shown in [link].



Solution:

We can introduce new input variable, t,representing years since 1994.

The least squares regression equation is:
Equation:

$$C(t) = 113.318 + 2.209t$$

Using technology, the correlation coefficient was calculated to be 0.9965, suggesting a very strong increasing linear trend.

Using this to predict consumption in 2008 $(t = 14)$,
Equation:

$$\begin{aligned} C(14) &= 113.318 + 2.209(14) \\ &= 144.244 \end{aligned}$$

The model predicts 144.244 billion gallons of gasoline consumption in 2008.

Key Concepts

- **Interpolation and extrapolation:**
  - The method of **interpolation** involves predicting a value inside the domain and/or range of the data.
  - The method of **extrapolation** involves predicting a value outside the domain and/or range of the data.
- **A proportion is an equation of the form** $\frac{a}{b} = \frac{c}{d}$, where $b \neq 0, d \neq 0$.
- **Univariate data has one variable and bivariate data has two variables**
- **Scatterplots:**
  - Scatter plots show the relationship between two sets of data. See [link].
  - Scatter plots may represent linear or non-linear models.
  - The line of best fit may be estimated or calculated, using a calculator or statistical software. See [link].
  - Interpolation can be used to predict values inside the domain and range of the data, whereas extrapolation can be used to predict values outside the domain and range of the data. See [link].
  - The correlation coefficient, r, indicates the degree of linear relationship between data. See [link].
  - A regression line best fits the data. See [link].

Glossary

correlation coefficient
: a value, r, between –1 and 1 that indicates the degree of linear correlation of variables, or how closely a regression line fits a data set.

least squares regression
: a statistical technique for fitting a line to data in a way that minimizes the differences between the line and data values

model breakdown
: when a model no longer applies after a certain point

Exponential Growth: Lessons 10.A - 10.B
By the end of this section, you will be able to:

- Graph exponential functions
- Solve Exponential equations
- Use exponential models in applications

This correlates to Lessons 10A-B from Corequisite MAT 1043(QR) and NCBO.

Note:
In case you missed something in class, or just want to review a specific topic covered in this Module, here is a list of topics covered:

1. Population Growth [link]
2. Defining Exponential Functions [link]
3. Identifying Exponential Functions [link]
4. Graphs of Exponential Functions [link]
5. Key Concepts [link]

The functions we have studied so far do not give us a model for many naturally occurring phenomena. From the growth of populations and the spread of viruses to radioactive decay and compounding interest, the models are very different from what we have studied so far. These models involve exponential functions.

Population Growth Rates [footnote]

This material was created by Amanda Towry using Microsoft Word processing program.

Population growth is often defined as exponential growth. **Exponential growth** is growth that is shown to be proportional to the size of the population in question.

Remember from [link] that exponential functions grow faster the larger the input value is. This means that the larger the population, the faster it will grow.

Not only will the population itself grow larger, but the rate at which it grows will also increase. This is well represented by exponential models. In order to determine if something is showing exponential growth, one should investigate if the rate of growth is proportional to the total population size.

The **rate of growth** of the population can be determined by the following equation:
Equation:

$$\text{Average Rate of Change in Popualtion} = \frac{\text{Change in Population}}{\text{Total Population}}$$

This average rate of change in the population is often written in symbols as:
Equation:

$$\text{Average Rate of Change in Popualtion} = \Delta P$$

When this ratio of change in population to change in time is proportional, or the same, over many time instances; it is considered to be a linear growth pattern. When the ratio is not proportional over time, it is not linear and one can investigate if it is exponential in form.

To determine if it is exponential, we must compare the previous ratio to the total popualtion at that point in time:
Equation:

$$\frac{\text{Average Rate of Change in Popualtion}}{\text{Total Popualtion}} = \frac{\frac{\text{Change in Population}}{\text{Change in Time}}}{\text{Total Population}}$$

This growth rate is often written in symbols as:
Equation:

$$\frac{\text{Average Rate of Change in Popualtion}}{\text{Total Popualtion}} = \frac{\Delta P}{P}$$

Example:
Linear or Exponential Growth?
Exercise:

Problem:

[link] shows how the population of a certain US city has changed over a few years. Determine if the growth is linear or exponential.

Year	Population (in thousands)
2000	2,020
2001	2,093
2002	2,168
2003	2,246
2004	2,327
2005	2,411
2006	2,498

Solution:

For each change in a year, find the average rate of change and divide it by the total population from the beginning year.

Years	Δ Population/Δ Time
2000-2001	$\frac{(2{,}093-2{,}020)}{1 \text{ year}} = 73$
2001-2002	$\frac{(2{,}168-2093)}{1 \text{ year}} = 75$
2002-2003	$\frac{(2{,}246-2{,}168)}{1 \text{ year}} = 78$
2003-2004	$\frac{(2{,}327-2{,}246)}{1 \text{ year}} = 81$
2004-2005	$\frac{(2{,}411-2{,}327)}{1 \text{ year}} = 84$
2005-2006	$\frac{(2{,}498-2{,}411)}{1 \text{ year}} = 87$

In order to determine if the data is linear growth, we look at the ratio outcomes in the second column of [link]. Notice that all the resultant ratios are not proportional; they're not the same ratio every time! This tells us that the data is **NOT** linear. Now let's check and see if the ratio of the average growth to total population is proportional.

Years	Δ Population/Δ Time	Avg. Population Growth/Total Population	Growth
2000-2001	$\frac{(2{,}093-2{,}020)}{1 \text{ year}} = 73$	$\frac{(73)}{2{,}020} = 0.0361$	3.61%
2001-2002	$\frac{(2{,}168-2093)}{1 \text{ year}} = 75$	$\frac{(75)}{2{,}020} = 0.0358$	3.58%
2002-2003	$\frac{(2{,}246-2{,}168)}{1 \text{ year}} = 78$	$\frac{(78)}{2{,}168} = 0.0359$	3.59%
2003-2004	$\frac{(2{,}327-2{,}246)}{1 \text{ year}} = 81$	$\frac{(81)}{2{,}246} = 0.0360$	3.60%
2004-2005	$\frac{(2{,}411-2{,}327)}{1 \text{ year}} = 84$	$\frac{(84)}{2{,}327} = 0.0361$	3.61%
2005-2006	$\frac{(2{,}498-2{,}411)}{1 \text{ year}} = 87$	$\frac{(87)}{2{,}411} = 0.0361$	3.61%

It can be seen from the third and fourth columns of [link] that the growth ratios are all very close to the value of 3.60%. Thus, we can conclude that this population data is indeed an example of exponential growth.

Defining Exponential Functions [footnote]

Section material derived from Openstax College Algebra: Exponential and Logarithmic Functions-Exponential Functions and Intermediate Algebra: Exponential and Logarithmic Functions-Evaluate and Graph Exponential Functions

Note:
For any real number x,an **exponential function** is a function with the form
Equation:

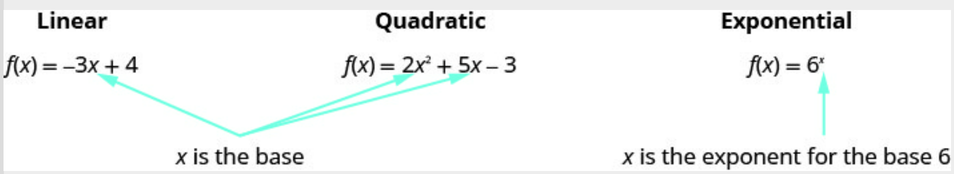
$$f(x) = ab^x$$

where

- a is a non-zero real number called the initial value and
- b is any positive real number such that $b \neq 1$.
- The domain of f is all real numbers.
- The range of f is all positive real numbers if $a > 0$.
- The range of f is all negative real numbers if $a < 0$.
- The y-intercept is $(0, a)$,and the horizontal asymptote is $y = 0$.

Notice that in this function, the variable is the exponent. In our functions so far, the variables were the base.
Our definition says $a \neq 1$. If we let $a = 1$, then $f\left(x\right) = a^x$ becomes $f\left(x\right) = 1^x$. Since $1^x = 1$ for all real numbers, $f\left(x\right) = 1$. This is the constant function.
Our definition also says $a > 0$. If we let a base be negative, say -4, then $f\left(x\right) = \left(-4\right)^x$ is not a real number when $x = \frac{1}{2}$.
In fact, $f\left(x\right) = \left(-4\right)^x$ would not be a real number any time x is a fraction with an even denominator. So our definition requires $a > 0$.
Equation:

$$\begin{array}{rcl} f\left(x\right) & = & \left(-4\right)^x \\ f\left(\frac{1}{2}\right) & = & \left(-4\right)^{\frac{1}{2}} \\ f\left(\frac{1}{2}\right) & = & \sqrt{-4} \quad \text{not a real number} \end{array}$$



Example:
Identifying Exponential Functions
Exercise:

Problem: Which of the following equations are *not* exponential functions?

- $f(x) = 4^{3(x-2)}$
- $g(x) = x^3$
- $h(x) = \left(\frac{1}{3}\right)^x$
- $j(x) = (-2)^x$

Solution:

By definition, an exponential function has a constant as a base and an independent variable as an exponent. Thus, $g(x) = x^3$ does not represent an exponential function because the base is an independent variable. In fact, $g(x) = x^3$ is a power function.

Recall that the base b of an exponential function is always a positive constant, and $b \neq 1$. Thus, $j(x) = (-2)^x$ does not represent an exponential function because the base, -2, is less than 0.

Identifying Exponential Functions [footnote]

Section material derived from Openstax College Algebra: Exponential and Logarithmic Functions-Exponential Functions

When exploring linear growth, we observed a constant rate of change—a constant number by which the output increased for each unit increase in input. For example, in the equation $f(x) = 3x + 4$,the slope tells us the output increases by 3 each time the input increases by 1. The scenario in the India population example is different because we have a *percent* change per unit time (rather than a constant change) in the number of people.

What exactly does it mean to *grow exponentially*? What does the word *double* have in common with *percent increase*? People toss these words around errantly. Are these words used correctly? The words certainly appear frequently in the media.

- **Percent change** refers to a *change* based on a *percent* of the original amount.
- **Exponential growth** refers to an *increase* based on a constant multiplicative rate of change over equal increments of time, that is, a *percent* increase of the original amount over time.
- **Exponential decay** refers to a *decrease* based on a constant multiplicative rate of change over equal increments of time, that is, a *percent* decrease of the original amount over time.

For us to gain a clear understanding of exponential growth, let us contrast exponential growth with linear growth. We will construct two functions. The first function is exponential. We will start with an input of 0, and increase each input by 1. We will double the corresponding consecutive outputs. The second function is linear. We will start with an input of 0, and increase each input by 1. We will add 2 to the corresponding consecutive outputs. See [link].

x	$f(x) = 2^x$	$g(x) = 2x$
0	1	0
1	2	2
2	4	4
3	8	6
4	16	8
5	32	10
6	64	12

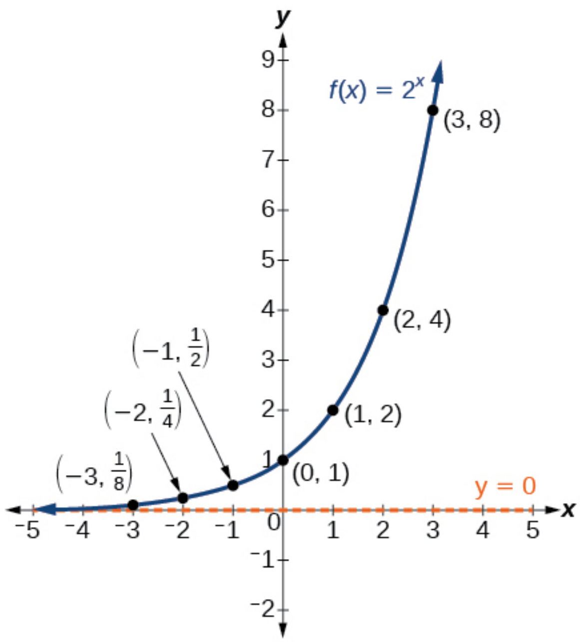
From [link] we can infer that for these two functions, exponential growth dwarfs linear growth.

- **Exponential growth** refers to the original value from the range increases by the **same percentage** over equal increments found in the domain.

- **Linear growth** refers to the original value from the range increases by the **same amount** over equal increments found in the domain.

Apparently, the difference between "the same percentage" and "the same amount" is quite significant. For exponential growth, over equal increments, the constant multiplicative rate of change resulted in doubling the output whenever the input increased by one. For linear growth, the constant additive rate of change over equal increments resulted in adding 2 to the output whenever the input was increased by one.

Let us examine the graph of f by plotting the ordered pairs we observe on the table in [link], and then make a few observations.



Let's define the behavior of the graph of the exponential function $f(x) = 2^x$ and highlight some its key characteristics.

- the domain is $(-\infty, \infty)$,
- the range is $(0, \infty)$,
- as $x \to \infty, f(x) \to \infty$,
- as $x \to -\infty, f(x) \to 0$,
- $f(x)$ is always increasing,
- the graph of $f(x)$ will never touch the *x*-axis because base two raised to any exponent never has the result of zero.
- $y = 0$ is the horizontal asymptote.
- the *y*-intercept is 1.

Graphs of Exponential Functions [footnote]

Section material derived from Openstax Intermediate Algebra: Exponential and Logarithmic Functions-Evaluate and Graph Exponential Functions and Precalculus: Exponential and Logarithmic Functions-Graphs of Exponential Functions

By graphing a few exponential functions, we will be able to see their unique properties.

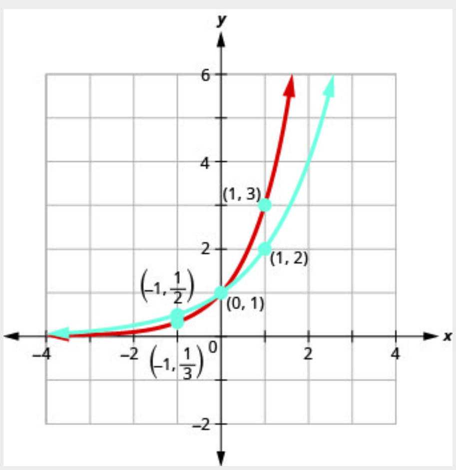
Example:
Graphing Exponential Functions
Exercise:

Problem: On the same coordinate system graph $f\left(x\right)={2}^{x}$ and $g\left(x\right)={3}^{x}$.

Solution:

We will use point plotting to graph the functions.

x	$f(x)=2^x$	$(x, f(x))$	$g(x)=3^x$	$(x, g(x))$
-2	$2^{-2}=\frac{1}{2^2}=\frac{1}{4}$	$\left(-2, \frac{1}{4}\right)$	$3^{-2}=\frac{1}{3^2}=\frac{1}{9}$	$\left(-2, \frac{1}{9}\right)$
-1	$2^{-1}=\frac{1}{2^1}=\frac{1}{2}$	$\left(-1, \frac{1}{2}\right)$	$3^{-1}=\frac{1}{3^1}=\frac{1}{3}$	$\left(-1, \frac{1}{3}\right)$
0	$2^0=1$	$(0, 1)$	$3^0=1$	$(0, 1)$
1	$2^1=2$	$(1, 2)$	$3^1=3$	$(1, 3)$
2	$2^2=4$	$(2, 4)$	$3^2=9$	$(2, 9)$
3	$2^3=8$	$(3, 8)$	$3^3=27$	$(3, 27)$



If we look at the graphs from the previous Example, we can identify some of the properties of exponential functions.

The graphs of $f\left(x\right)={2}^{x}$ and $g\left(x\right)={3}^{x}$ have the same basic shape. This is the shape we expect from an exponential function where $a>1$.

We notice, that for each function, the graph contains the point $\left(0,1\right)$. This make sense because ${a}^{0}=1$ for any a.

The graph of each function, $f\left(x\right)={a}^{x}$ also contains the point $\left(1,a\right)$. The graph of $f\left(x\right)={2}^{x}$ contained $\left(1,2\right)$ and the graph of $g\left(x\right)={3}^{x}$ contained $\left(1,3\right)$. This makes sense as ${a}^{1}=a$.

Notice too, the graph of each function $f\left(x\right)={a}^{x}$ also contains the point $\left(-1,\frac{1}{a}\right)$. The graph of $f\left(x\right)={2}^{x}$ contained $\left(-1,\frac{1}{2}\right)$ and the graph of $g\left(x\right)={3}^{x}$ contained $\left(-1,\frac{1}{3}\right)$. This makes sense as ${a}^{-1}=\frac{1}{a}$.

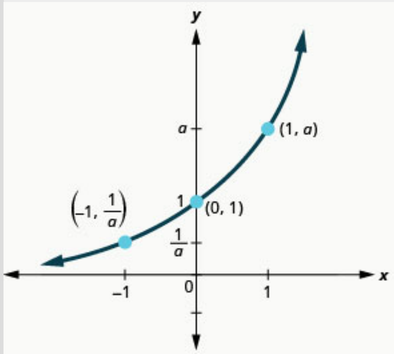
What is the domain for each function? From the graphs we can see that the domain is the set of all real numbers. There is no restriction on the domain. We write the domain in interval notation as $(-\infty, \infty)$.

Look at each graph. What is the range of the function? The graph never hits the x-axis. The range is all positive numbers. We write the range in interval notation as $(0, \infty)$.

Whenever a graph of a function approaches a line but never touches it, we call that line an **asymptote**. For the exponential functions we are looking at, the graph approaches the x-axis very closely but will never cross it, we call the line $y = 0$, the *x*-axis, a horizontal asymptote.

Note:

Domain	$(-\infty, \infty)$
Range	$(0, \infty)$
x-intercept	None
y-intercept	$(0, 1)$
Contains	$(1, a)$, $\left(-1, \frac{1}{a}\right)$
Asymptote	x-axis, the line $y = 0$



Our definition of an exponential function $f(x) = a^x$ says $a > 0$, but the examples and discussion so far has been about functions where $a > 1$. What happens when $0 < a < 1$? The next example will explore this possibility.

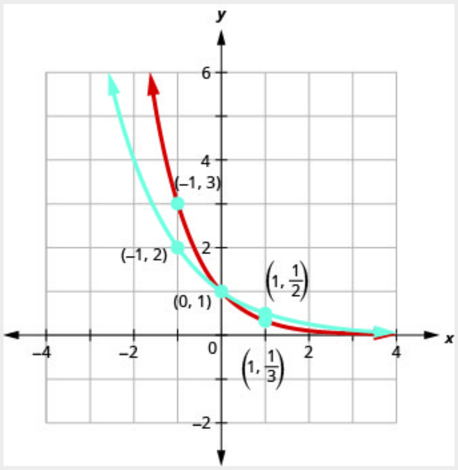
Example:
Graphing Exponential Functions
Exercise:

Problem: On the same coordinate system, graph $f(x) = \left(\frac{1}{2}\right)^x$ and $g(x) = \left(\frac{1}{3}\right)^x$.

Solution:

We will use point plotting to graph the functions.

x	$f(x) = \left(\frac{1}{2}\right)^x$	$(x, f(x))$	$g(x) = \left(\frac{1}{3}\right)^x$	$(x, g(x))$
-2	$\left(\frac{1}{2}\right)^{-2} = 2^2 = 4$	$(-2, 4)$	$\left(\frac{1}{3}\right)^{-2} = 3^2 = 9$	$(-2, 9)$
-1	$\left(\frac{1}{2}\right)^{-1} = 2^1 = 2$	$(-1, 2)$	$\left(\frac{1}{3}\right)^{-1} = 3^1 = 3$	$(-1, 3)$
0	$\left(\frac{1}{2}\right)^0 = 1$	$(0, 1)$	$\left(\frac{1}{3}\right)^0 = 1$	$(0, 1)$
1	$\left(\frac{1}{2}\right)^1 = \frac{1}{2}$	$\left(1, \frac{1}{2}\right)$	$\left(\frac{1}{3}\right)^1 = \frac{1}{3}$	$\left(1, \frac{1}{3}\right)$
2	$\left(\frac{1}{2}\right)^2 = \frac{1}{4}$	$\left(2, \frac{1}{4}\right)$	$\left(\frac{1}{3}\right)^2 = \frac{1}{9}$	$\left(2, \frac{1}{9}\right)$
3	$\left(\frac{1}{2}\right)^3 = \frac{1}{8}$	$\left(3, \frac{1}{8}\right)$	$\left(\frac{1}{3}\right)^3 = \frac{1}{27}$	$\left(3, \frac{1}{27}\right)$



Now let's look at the graphs from the previous Example so we can now identify some of the properties of exponential functions where $0 < a < 1$.

The graphs of $f(x) = \left(\frac{1}{2}\right)^x$ and $g(x) = \left(\frac{1}{3}\right)^x$ have the same basic shape. While this is the shape we expect from an exponential function where $0 < a < 1$, the graphs go down from left to right while the previous graphs, when $a > 1$, went from up from left to right.

We notice that for each function, the graph still contains the point (0, 1). This make sense because $a^0 = 1$ for any a.

As before, the graph of each function, $f\left(x\right)=a^{x}$, also contains the point $\left(1,a\right)$. The graph of $f\left(x\right)=\left(\frac{1}{2}\right)^{x}$ contained $\left(1,\frac{1}{2}\right)$ and the graph of $g\left(x\right)=\left(\frac{1}{3}\right)^{x}$ contained $\left(1,\frac{1}{3}\right)$. This makes sense as $a^{1}=a$.

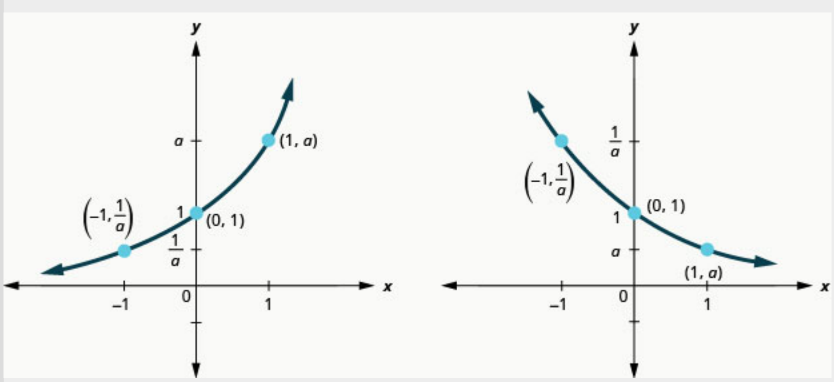
Notice too that the graph of each function, $f\left(x\right)=a^{x}$, also contains the point $\left(-1,\frac{1}{a}\right)$. The graph of $f\left(x\right)=\left(\frac{1}{2}\right)^{x}$ contained $\left(-1,2\right)$ and the graph of $g\left(x\right)=\left(\frac{1}{3}\right)^{x}$ contained $\left(-1,3\right)$. This makes sense as $a^{-1}=\frac{1}{a}$.

What is the domain and range for each function? From the graphs we can see that the domain is the set of all real numbers and we write the domain in interval notation as $\left(-\infty,\infty\right)$. Again, the graph never hits the x-axis. The range is all positive numbers. We write the range in interval notation as $\left(0,\infty\right)$.

We will summarize these properties in the chart below. Which also include when $a>1$.

Note:

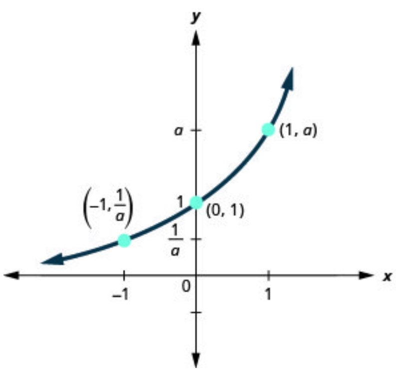
when $a>1$		**when** $0<a<1$	
Domain	$\left(-\infty,\infty\right)$	Domain	$\left(-\infty,\infty\right)$
Range	$\left(0,\infty\right)$	Range	$\left(0,\infty\right)$
x-intercept	none	x-intercept	none
y-intercept	$\left(0,1\right)$	y-intercept	$\left(0,1\right)$
Contains	$\left(1,a\right),\ \left(-1,\frac{1}{a}\right)$	Contains	$\left(1,a\right),\ \left(-1,\frac{1}{a}\right)$
Asymptote	x-axis, the line $y=0$	Asymptote	x-axis, the line $y=0$
Basic shape	increasing	Basic shape	decreasing

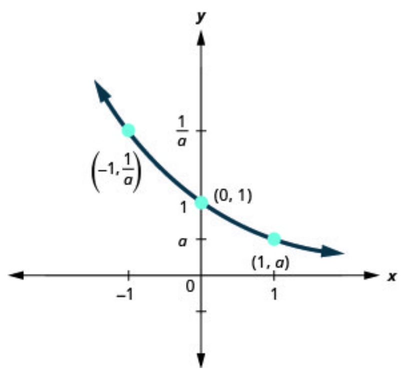


Key Concepts

- **Properties of the Graph of $f(x) = a^x$:**

when $a > 1$		**when $0 < a < 1$**	
Domain	$(-\infty, \infty)$	Domain	$(-\infty, \infty)$
Range	$(0, \infty)$	Range	$(0, \infty)$
x-intercept	none	x-intercept	none
y-intercept	$(0, 1)$	y-intercept	$(0, 1)$
Contains	$(1, a)$, $\left(-1, \frac{1}{a}\right)$	Contains	$(1, a)$, $\left(-1, \frac{1}{a}\right)$
Asymptote	x-axis, the line $y = 0$	Asymptote	x-axis, the line $y = 0$
Basic shape	increasing	Basic shape	decreasing





- **Average rate of growth in a popualtion**

$$\textit{Average Rate of Change in Population} = \frac{\textit{Change in Population}}{\textit{Change in Time}}$$

- **To determine if a rate of growth is exponential**

$$\frac{Average\ Rate\ of\ Change\ in\ Population}{Total\ Popualtion} = \frac{\frac{Change\ in\ Population}{Change\ in\ Time}}{Total\ Popualtion}$$

Glossary

Asymptote

A line which a graph of a function approaches closely but never touches.

Exponential Function

An exponential function, where $a > 0$ and $a \neq 1$, is a function of the form $f(x) = a^x$.

Exponential Growth

A rate of growth that increases in proportion to the original population.

Linear Growth

A rate of growth that increases in the exact same amount for each step in time.

Rate of Growth

The rate at which a popualtion increases over time.

Logistic Growth: Lessons 11.A - 11.E
By the end of this section, you will be able to:

- Explain the characteristics of and differences between exponential and logistic growth patterns
- Give examples of exponential and logistic growth in natural populations
- Give examples of how the carrying capacity of a habitat may change
- Compare and contrast density-dependent growth regulation and density-independent growth regulation giving examples

This correlates to Lessons 11A-E of Corequisite MAT 1043(QR) and NCBO.

Note:
Topics Covered
In case you missed something in class, or just want to review a specific topic covered in this Module, here is a list of topics covered:

1. Exponential Population Growth [link]
2. Logistic Population Growth[link]
3. Key Concepts [link]

Population ecologists make use of a variety of methods to model population dynamics. An accurate model should be able to describe the changes occurring in a population and predict future changes.

The two simplest models of population growth use deterministic equations (equations that do not account for random events) to describe the rate of change in the size of a population over time. The first of these models, exponential growth, describes theoretical populations that increase in numbers without any limits to their growth. The second model, logistic growth, introduces limits to reproductive growth that become more intense

as the population size increases. Neither model adequately describes natural populations, but they provide points of comparison.

Exponential Population Growth [footnote]

Section material derived from Openstax Concepts of Biology: Ecology-Populations and Community Ecology-Population Growth and Regulation

Charles Darwin, in developing his theory of natural selection, was influenced by the English clergyman Thomas Malthus. Malthus published his book in 1798 stating that populations with abundant natural resources grow very rapidly; however, they limit further growth by depleting their resources. The early pattern of accelerating population size is called **exponential growth**.

The best example of exponential growth in organisms is seen in bacteria. Bacteria are prokaryotes that reproduce largely by binary fission. This division takes about an hour for many bacterial species. If 1000 bacteria are placed in a large flask with an abundant supply of nutrients (so the nutrients will not become quickly depleted), the number of bacteria will have doubled from 1000 to 2000 after just an hour. In another hour, each of the 2000 bacteria will divide, producing 4000 bacteria. After the third hour, there should be 8000 bacteria in the flask. The important concept of exponential growth is that the growth rate—the number of organisms added in each reproductive generation—is itself increasing; that is, the population size is increasing at a greater and greater rate. After 24 of these cycles, the population would have increased from 1000 to more than 16 billion bacteria. When the population size, N, is plotted over time, a **J-shaped growth curve** is produced ([link]**a**).

The bacteria-in-a-flask example is not truly representative of the real world where resources are usually limited. However, when a species is introduced into a new habitat that it finds suitable, it may show exponential growth for a while. In the case of the bacteria in the flask, some bacteria will die during the experiment and thus not reproduce; therefore, the growth rate is lowered from a maximal rate in which there is no mortality. The growth rate of a population is largely determined by subtracting the **death rate**, D, (number

organisms that die during an interval) from the **birth rate**, *B*, (number organisms that are born during an interval). The growth rate can be expressed in a simple equation that combines the birth and death rates into a single factor: *k*. This is shown in the following formula:

Equation:

$$\text{Population growth } = (B - D)P$$

Equation:

$$\text{Population growth } = kP$$

The value of *k* can be positive, meaning the population is increasing in size (the rate of change is positive); or negative, meaning the population is decreasing in size; or zero, in which case the population size is unchanging, a condition known as **zero population growth**.

Logistic Population Growth [footnote]

Section material derived from Openstax Concepts of Biology: Ecology-Populations and Community Ecology-Population Growth and Regulation

Extended exponential growth is possible only when infinite natural resources are available; this is not the case in the real world. Charles Darwin recognized this fact in his description of the "struggle for existence," which states that individuals will compete (with members of their own or other species) for limited resources. The successful ones are more likely to survive and pass on the traits that made them successful to the next generation at a greater rate (natural selection). To model the reality of limited resources, population ecologists developed the **logistic growth** model.

Carrying Capacity and the Logistic Model

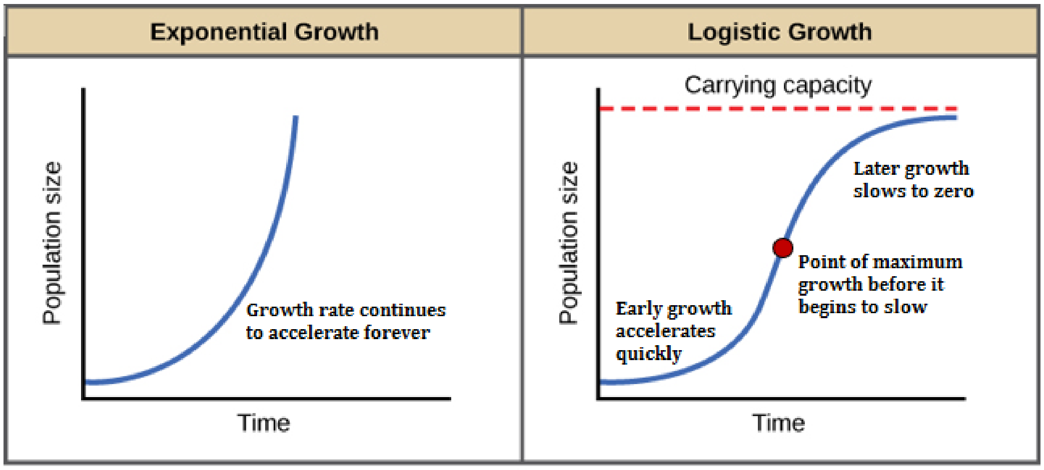
In the real world, with its limited resources, exponential growth cannot continue indefinitely. Exponential growth may occur in environments where there are few individuals and plentiful resources, but when the number of individuals gets large enough, resources will be depleted and the growth rate will slow down. Eventually, the growth rate will plateau or level off ([link]**b**). This population size, which is determined by the maximum population size that a particular environment can sustain, is called the **carrying capacity**, or *L*. In real populations, a growing population often overshoots its carrying capacity, and the death rate increases beyond the birth rate causing the population size to decline back to the carrying capacity or below it. Most populations usually fluctuate around the carrying capacity in an undulating fashion rather than existing right at it.

The formula used to calculate logistic growth adds the carrying capacity as a moderating force in the growth rate. The expression "*L* – *P*" is equal to the number of individuals that may be added to a population at a given time, and "*L* – *P*" divided by "*L*" is the fraction of the carrying capacity available for further growth. Thus, the exponential growth model is restricted by this factor to generate the logistic growth equation:

Equation:

$$\text{Population growth} = kP\left[\frac{L-P}{L}\right]$$

Notice that when *P* is almost zero the quantity in brackets is almost equal to 1 (or *L*/*L*) and growth is close to exponential. When the population size is equal to the carrying capacity, or *P* = *L*, the quantity in brackets is equal to zero and growth is equal to zero. A graph of this equation (logistic growth) yields the **S-shaped curve** ([link]**b**). It is a more realistic model of population growth than exponential growth. There are three different sections to an S-shaped curve. Initially, growth is exponential because there are few individuals and ample resources available. Then, as resources begin to become limited, the growth rate decreases. Finally, the growth rate levels off at the carrying capacity of the environment, with little change in population number over time.



(a) (b)

When resources are unlimited, populations exhibit (a) exponential growth, shown in a J-shaped curve. When resources are limited, populations exhibit (b) logistic growth. In logistic growth, population expansion decreases as resources become scarce, and it levels off when the carrying capacity of the environment is reached. The logistic growth curve is S-shaped.

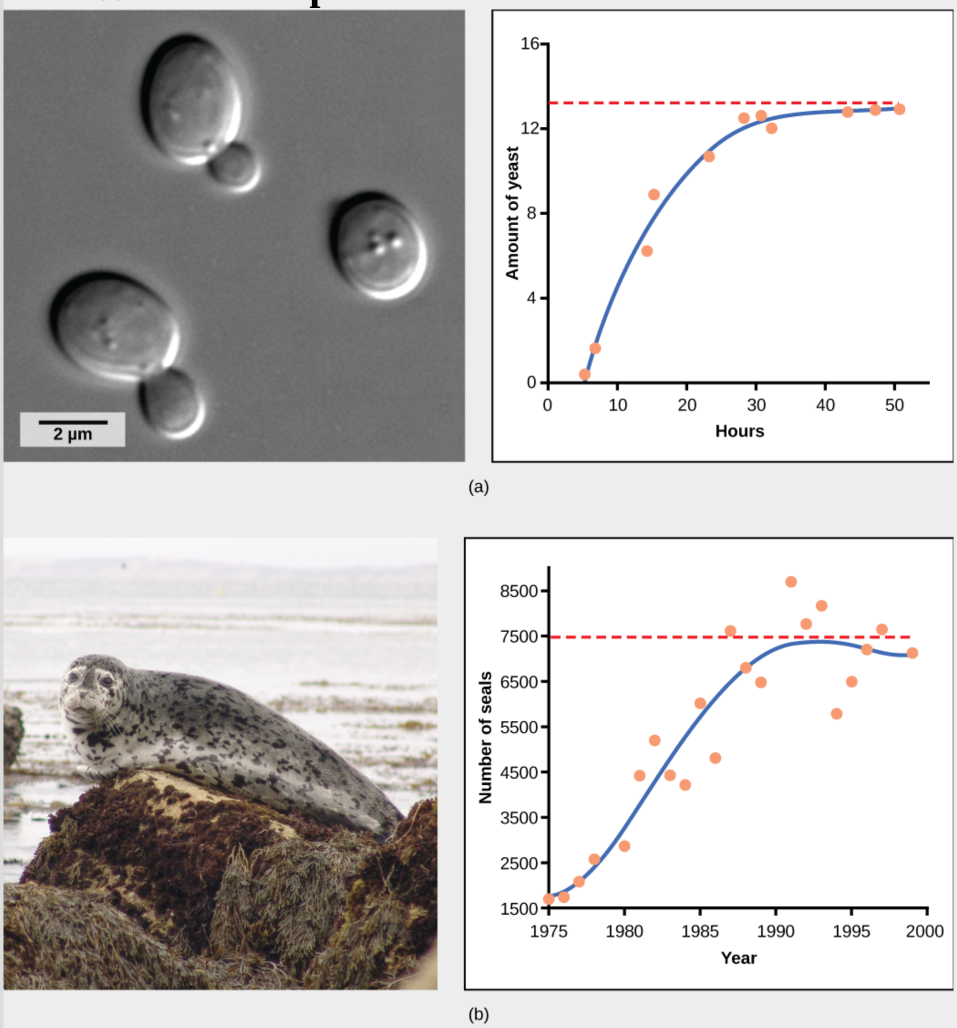
Examples of Logistic Growth [footnote]

Section material derived from Openstax Concepts of Biology: Ecology-Populations and Community Ecology-Population Growth and Regulation

Yeast, a microscopic fungus used to make bread and alcoholic beverages, exhibits the classical S-shaped curve when grown in a test tube ([link]**a**). Its growth levels off as the population depletes the nutrients that are necessary for its growth. In the real world, however, there are variations to this idealized curve. Examples in wild populations include sheep and harbor seals ([link]**b**). In both examples, the population size exceeds the carrying

capacity for short periods of time and then falls below the carrying capacity afterwards. This fluctuation in population size continues to occur as the population oscillates around its carrying capacity. Still, even with this oscillation, the logistic model is confirmed.

Example:
Real World Examples



(a) Yeast grown in ideal conditions in a test tube shows a classical S-shaped logistic growth curve, whereas (b) a natural population of seals shows real-world fluctuation. The yeast is visualized using

differential interference contrast light micrography.
(credit a: scale-bar data from Matt Russell)

Key Concepts

- **Logistic population growth**
  Equation:

$$\text{Population growth} = kP\left[\frac{L-P}{L}\right]$$

Glossary

Birth Rate
: the number of births within a population at a specific point in time

Carrying Capacity
: the maximum number of individuals of a population that can be supported by the limited resources of a habitat

Death Rate
: the number of deaths within a population at a specific point in time

Exponential Growth
: an accelerating growth pattern seen in populations where resources are not limiting

J-shaped growth curve
: the shape of an exponential growth curve

Logistic Growth
: the leveling off of exponential growth due to limiting resources

S-shaped growth curve

the shape of a logistic growth curve

Zero Population Growth

the steady population size where birth rates and death rates are equal

Sinusoidal Functions: 12.A - 12.B
In this section, you will:

- Graph variations of $y = \sin(x)$ and $y = \cos(x)$.
- Use phase shifts of sine and cosine curves.

This correlates to Lessons 12A-B of Corequisite MAT 1043(QR) and NCBO.

Note:
In case you missed something in class, or just want to review a specific topic covered in this Module, here is a list of topics covered:

1. The Sine Function [link]
2. Determining the Period of a sinusoidal Function [link]
3. Determining the Amplitude [link]
4. Key Concepts [link]

White light, such as the light from the sun, is not actually white at all. Instead, it is a composition of all the colors of the rainbow in the form of waves. The individual colors can be seen only when white light passes through an optical prism that separates the waves according to their wavelengths to form a rainbow. Light waves can be represented graphically by the sine function.

Sinusoidal Functions: The Sine Function [footnote]

Section material derived from Openstax Precalculus: Periodic Functions-Gaphs of the Sine and Cosine Functions

As we can see, sine functions have a regular period and range. If we watch ocean waves or ripples on a pond, we will see that they resemble the sine function. However, they are not necessarily identical. Some are taller or

longer than others. A function that has the same general shape as a **sine function** is known as a **sinusoidal function**. The general forms of sinusoidal functions are

Equation:

$$y = A\sin\left(Bx - C\right) + D$$

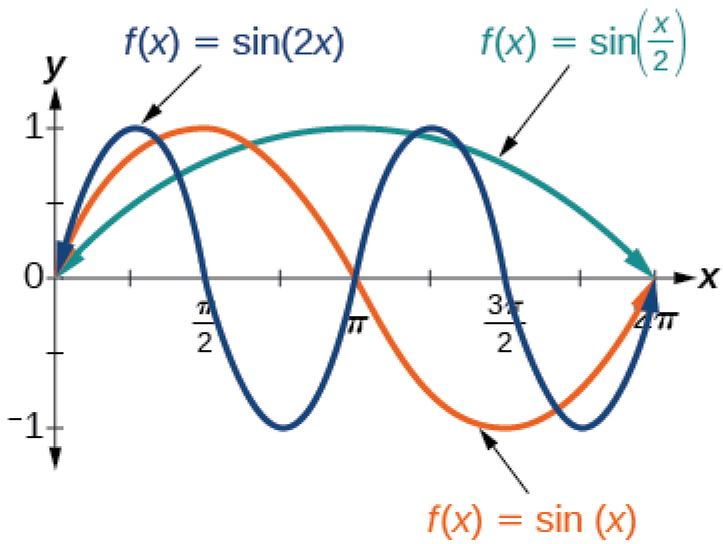
where, A is the amplitude, B is the variable that determines the period, C specifies any horizontal shifts, and D specifies any vertical shifts.

Determining the Period of Sinusoidal Functions [footnote]

Section material derived from Openstax Precalculus: Periodic Functions- Gaphs of the Sine and Cosine Functions

Looking at the forms of sinusoidal functions, we can see that they are transformations of the sine and cosine functions. We can use what we know about transformations to determine the period.

In the general formula, B is related to the period by $P = \frac{2\pi}{|B|}$. If $|B| > 1$, then the period is less than 2π and the function undergoes a horizontal compression, whereas if $|B| < 1$, then the period is greater than 2π and the function undergoes a horizontal stretch. For example, $f(x) = \text{sin}(x\ )$, $B = 1$, so the period is 2π,which we knew. If $f(x) = \sin\left(2x\right)$, then $B = 2$, so the period is π and the graph is compressed. If $f(x) = \sin\left(\frac{x}{2}\right)$, then $B = \frac{1}{2}$, so the period is 4π and the graph is stretched. Notice in [link] how the period is indirectly related to $|B|$.



Note:
Period of Sinusoidal Functions
If we let $C = 0$ and $D = 0$ in the general form equations of the sine and cosine functions, we obtain the forms
Equation:

$$y = A \sin(Bx)$$

The period is $\frac{2\pi}{|B|}$.

Determining the Amplitude of Sinusoidal Functions [footnote]

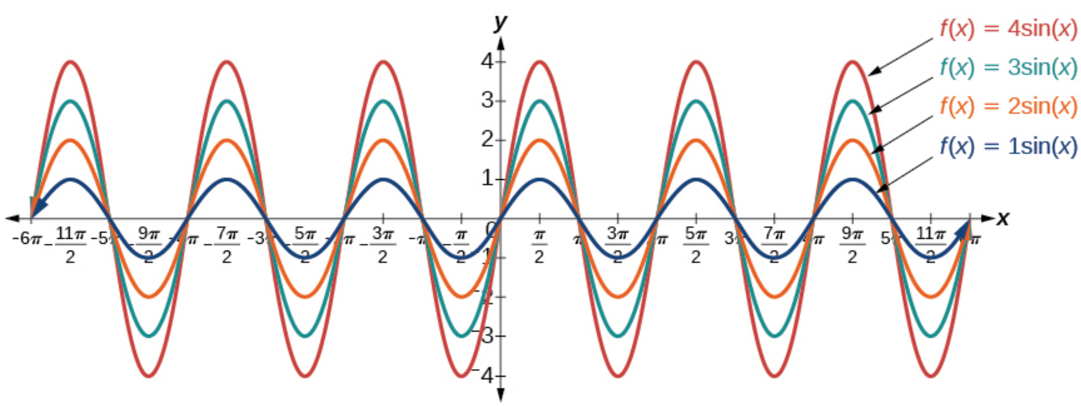
Section material derived from Openstax Precalculus: Periodic Functions-Gaphs of the Sine and Cosine Functions

Returning to the general formula for a sinusoidal function, we have analyzed how the variable B relates to the period. Now let's turn to the variable A so we can analyze how it is related to the **amplitude**, or greatest distance from rest. A represents the vertical stretch factor, and its absolute value $|A|$ is the amplitude. The local maxima will be a distance $|A|$ above

the horizontal **midline** of the graph, which is the line $y = D$; because $D = 0$ in this case, the midline is the *x*-axis. The local minima will be the same distance below the midline. If $|A| > 1$, the function is stretched. For example, the amplitude of $f(x) = 4 \sin x$ is twice the amplitude of

$$f(x) = 2 \sin x.$$

If $|A| < 1$, the function is compressed. [link] compares several sine functions with different amplitudes.



Note:
Amplitude of Sinusoidal Functions
If we let $C = 0$ and $D = 0$ in the general form equation of the sine function, we obtain the form
Equation:

$$y = A \sin (Bx)$$

The **amplitude** is A, and the vertical height from the **midline** is $|A|$. In addition, notice in the example that
Equation:

$$|A| = \text{amplitude} = \frac{1}{2}|\text{maximum} - \text{minimum}|$$

Example:
Identifying the Amplitude of a Sine Function
Exercise:

Problem:

What is the amplitude of the sinusoidal function $f(x) = -4\sin(x)$? Is the function stretched or compressed vertically?

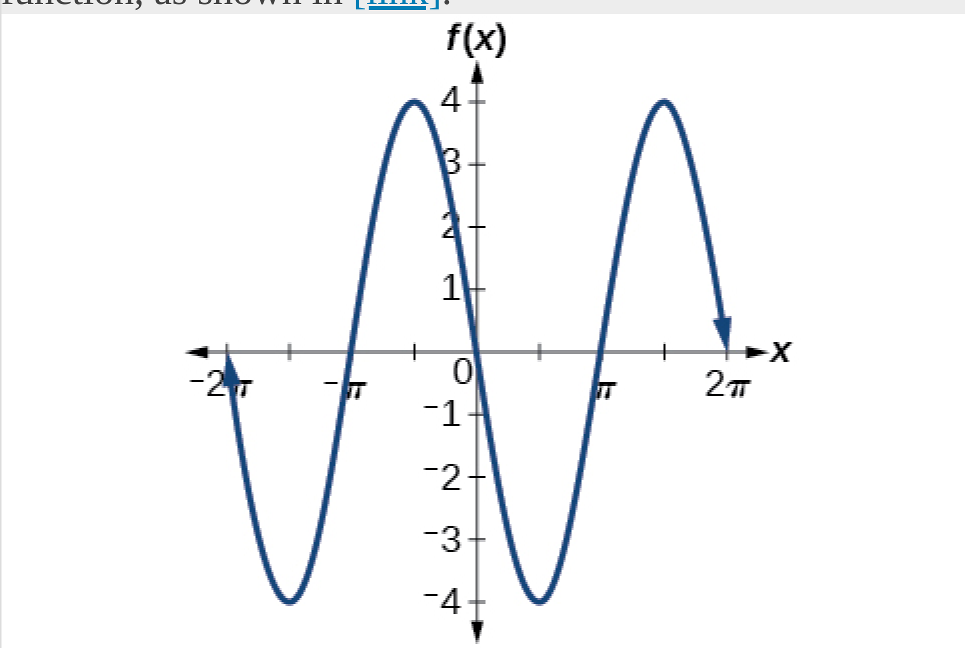
Solution:

Let's begin by comparing the function to the simplified form $y = A\sin(Bx)$.

In the given function, $A = -4$, so the amplitude is $|A| = |-4| = 4$. The function is stretched.

Analysis

The negative value of A results in a reflection across the *x*-axis of the sine function, as shown in [link].



Key Concepts

- The sine function has the equation
  Equation:
  $$f\left(x\right)=A\sin\left(Bx-C\right)+D$$
  where, A is the amplitude, B is the variable that determines the period, C specifies any horizontal shifts, and D specifies any vertical shifts.

Key Concepts

- Periodic functions repeat after a given value. The smallest such value is the period. The basic sine function has a period of 2π.
- The function $\sin\ x$ is odd, so its graph is symmetric about the origin.
- The graph of a sinusoidal function has the same general shape as a sine or cosine function.
- In the general formula for a sinusoidal function, the period is $P=\frac{2\pi}{|B|}$. See [link].
- In the general formula for a sinusoidal function, $|A|$ represents amplitude. If $|A|>1$, the function is stretched, whereas if $|A|<1$, the function is compressed. See [link].
- The value D in the general formula for a sinusoidal function indicates the vertical shift from the midline. See [link].
- The equation for a sinusoidal function can be determined from a graph. See [link] and [link].
- A function can be graphed by identifying its amplitude and period. See [link] and [link].
- Sinusoidal functions can be used to solve real-world problems. See [link], [link], and [link].

Glossary

amplitude
: the vertical height of a function; the constant A appearing in the definition of a sinusoidal function

midline
: the horizontal line $y = D$, where D appears in the general form of a sinusoidal function

periodic function
: a function $f(x)$ that satisfies $f(x + P) = f(x)$ for a specific constant P and any value of x

sinusoidal function
: any function that can be expressed in the form $f(x) = A\sin(Bx - C) + D$ or $f(x) = A\cos(Bx - C) + D$